Submitted to *Management Science*
manuscript MS-0001-1922.65

Sample-Based Online Generalized Assignment Problem with Unknown Poisson Arrivals

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We study an edge-weighted online stochastic *Generalized Assignment Problem* with *unknown* Poisson arrivals. In this model, we consider a bipartite graph that contains offline bins and online items, where each offline bin is associated with a D-dimensional capacity vector and each online item is with a D-dimensional demand vector which may be different towards each bin. Online arrivals are sampled from a set of online item types which follow independent but not necessarily identical Poisson processes. The arrival rate for each Poisson process is unknown. Each online item will either be packed into an offline bin which will deduct the allocated bin's capacity vector and generate a reward, or be rejected. The decision should be made immediately and irrevocably upon its arrival. Our goal is to maximize the total reward of the allocation without violating the capacity constraints. We provide a sample-based multi-phase algorithm by utilizing both pre-existing offline data (named historical data) and sequentially revealed online data. The developed algorithm employs the concept of exploration-exploitation to dynamically learn the arrival rate and optimize the allocation decision. We establish its parametric performance guarantee measured by a competitive ratio. We further provide a guideline on fine tuning the parameters under different sizes of historical data based on the established parametric form. By analyzing a special case which is a classical online weighted matching problem, we also provide a novel insight on how the historical data's quantity and quality (measured by the number of underrepresented agents in the data) affect the trade-off between exploration and exploitation in online algorithms and their performance. Finally, we demonstrate the effectiveness of our algorithms numerically.

Key words: Sample-based Algorithm, Online Resource Allocation, Competitive Ratio

1. Introduction

Online Generalized Assignment Problem (GAP) is a fundamental research topic in resource allocation, which has been widely studied in various areas including operations research and computer science (Alaei et al. 2013, Naori and Raz 2019, Albers et al. 2020, Jiang et al.

2021). In this problem, we are given a set of offline bins with general capacities. During the online process, online items arrive in the system sequentially and request a certain bin capacity, name demand. Capacity and demand are in multi-dimensions and they can be different for different resources and requests. We need to pack the item into an offline bin with enough remaining capacity immediately and irrevocably upon its arrival or reject the item. If an item is packed into a bin, it consumes the bin's capacity by its demand and generates a reward. Our goal is to maximize the total reward of the output packing scheme.

This model has wide applications in various domains. Examples include online task assignment, ridesharing, and cloud computing.

Online task assignment. In some online task assignment platforms such as Amazon, the platform has several tasks to be allocated to a sequence of online arriving workers. Each task can be regarded as an offline bin and contains several processes. Each worker is regarded as an item with a single demand. Each worker arrives at the platform sequentially and takes one process in a task upon its arrival. A successful match of a worker to a task generates a reward that depends on the quality of the task completed by this worker.

Ridesharing. In a ridesharing platform such as Uber, each car can be regarded as a bin whose capacity is the number of guest seats in the car. Each ride request arrives at the platform in an online manner and requests some seats from a car nearby. A car can take multiple ride requests if the capacity permits. A successful match between a ride request and a car will generate a certain reward that depends on the car's location and the passenger's destination.

Cloud computing. In some commercial cloud computing services such as Azure and Google Cloud, multiple servers with different configurations are provided to agents. Agents arrive sequentially and request a certain amount of computing resources such as CPU and memory from a server. Allocating different servers to an agent may lead to different rewards depending on the servers' configurations. The services provider's goal is to maximize the total reward.

The general framework of the problem lends itself to a wide range of applications, but it also poses challenges when it comes to devising an efficient algorithm. Aggarwal et al. (2011) demonstrated that no online algorithm can achieve a positive competitive ratio if the order of arrivals is determined by an adversary. However, if we assume that the arrivals follow a random order model, where the arriving order of online items is uniformly chosen from all possible permutations, Albers et al. (2020) presented an online algorithm for the single-dimensional GAP that achieves a competitive ratio of $\frac{1}{6.99}$. However, there is a scarcity of literature on the multi-dimensional GAP. To the best of our knowledge, Naori and Raz (2019) is the first one to provide an algorithm with a competitive ratio that depends on the dimensions under the random order arrival model. But there is no existing literature on the multi-dimensional GAP under a stochastic arrival model.

Poisson arrival is a widely-used stochastic arrival model. It has been extensively examined in various online problems (see Gamarnik and Squillante (2005), Jiang et al. (2021), Yan (2022)). In a typical Poisson arrival model, each online arrival of type $v \in V$ follows a Poisson process with a known arrival rate λ_v. The arrival processes of different types are assumed to be independent. However, in many real-world scenarios, specific information regarding the arrival rates is often unavailable. In other words, the Poisson arrival rate for each type is unknown.

In this paper, we study an online stochastic multidimensional generalized assignment (GAP) problem in the context of an unknown Poisson arrival model. This study marks the initial endeavor to explore the online GAP in an unknown Poisson arrival setting. To tackle this problem, we leverage both the pre-existing offline data, referred to as historical data, and the sequentially-revealed online data, and develop an efficient multi-phase algorithm. Specifically, the algorithm includes a sampling phase to reject all arrivals and only collect arrival data, as well as several exploitation phases to allocate resources to online arrivals. The developed algorithm employs the concept of exploration-exploitation to dynamically learn the arrival rate and optimize the allocation decision. Through a thorough analysis of the algorithm's performance, we examine the effect of the historical data size on its effectiveness. We first prove that even in the absence of historical data, our algorithm can generate a better ratio than that achieved by Naori and Raz (2019), the sole existing literature on online multidimensional GAP. Second, we provide a guideline for fine tuning the parameters when different sizes of historical data are available. Specifically, we introduce two heuristic algorithms derived from our main algorithm by selectively skipping certain phases. These heuristic algorithms are accompanied by established performance guarantees with fined tuned parameters. Furthermore, we conduct a thorough analysis of their performance across different levels of data availability. Finally, by restricting the models to a special case where the dimension $D = 1$ and both the capacity of offline bins and the demand of online items are set to 1, the problem reduces to a classical online weighted matching problem. By analyzing the competitive ratio for this online matching problem, we provide a novel insight on how the historical data's quantity and quality (measured by the number of underrepresented agents in the data) affect the trade-off between exploration and exploitation in online algorithms and their performance.

1.1. Related Work

Extensive research has been conducted on the online generalized assignment problem, along with its various simplified versions like online bin packing and online knapsack problems, as evidenced by several studies in the literature (Alaei et al. 2013, Naori and Raz 2019, Albers et al. 2020, Jiang et al. 2021).

A significant portion of the literature has primarily focused on single-dimensional GAP. Albers et al. (2020) proposed a randomized algorithm that achieves a competitive ratio of $\frac{1}{6.99}$ when considering the random order model. In a special case where there is only one offline bin, their algorithm can achieve a competitive ratio of $\frac{1}{6.65}$ for the online knapsack problem. Jiang et al. (2021) studied the case where the arrivals are drawn from some independent (but not identical) distributions over item types, and proposed a reduction technique that reduces the single-dimensional GAP to a sequence of online knapsack problems. The developed algorithm achieves a performance guarantee of $\frac{1}{3+e^2} \approx 0.319$ in both single-dimensional GAP and online knapsack problem. Alaei et al. (2013) studied a random demand model, i.e., the demand of each online item is also a random variable. They assume the demand realization can only be observed after the bin packing decision has been made. Assuming that the demand of each item does not exceed a fraction of $\frac{1}{k}$ of any bin's capacity, and given advance knowledge of the distribution information for all potential arrivals, they presented an algorithm. This algorithm achieves a competitive ratio of $1 - \frac{1}{\sqrt{k}}$ when items follow a stochastic arrival model.

If we further restrict the setting to the special case where both the demand and capacity are unitary under the dimension $D = 1$, the problem simplifies to an online bipartite matching problem. Extensive research has been conducted on the online bipartite matching problem in the literature. Karp et al. (1990) started this stream of works and considered maximizing the total number of matches under the worst-case model. They presented an algorithm with a tight competitive ratio of $1-\frac{1}{e}$. Many follow-ups further study maximizing the vertex-weighted or edge-weighted matching under a random order arrival model or a stochastic arrival model (Feldman et al. 2009, Aggarwal et al. 2011, Kesselheim et al. 2013, Huang and Shu 2021, Huang et al. 2022, Yan 2022, Feng et al. 2023). In particular, for maximizing the edge-weighted matching under the stochastic arrival model, Feldman et al. (2009) proposed a linear-program-based Suggest Matching algorithm and showed that this algorithm can achieve a competitive ratio of $1-\frac{1}{e}$. Their algorithm provides some intuitions for us in our algorithm design. Recently, Huang et al. (2022) proposed a state-of-the-art algorithm under the vertex-weighted setting which achieves a competitive ratio of 0.716 and Feng et al. (2023) proposed an algorithm with a competitive ratio of 0.650 under the edge-weighted setting when all arrivals follow a stochastic arrival process. Under the random order model, Kesselheim et al. (2013) presented an algorithm that achieves a competitive ratio of $\frac{1}{e}$, and Mehta (2013) showed the tightness of this bound.

However, there has been limited literature on the multi-dimensional GAP. Naori and Raz (2019) is a seminar work. It is also most related to our paper. They studied a random order model and proved that there exists a $O(D)$-competitive algorithms in the D-dimensional GAP, i.e., the competitive ratio of their algorithm is in the order of $\frac{1}{D}$. They also proved that the bound is tight in the order. In contrast, our research focuses on investigating the multi-dimensional generalized assignment problem (GAP) within a stochastic arrival model. More specifically, we assume the arrival model follows a Poisson arrival process with an unknown rate. Our algorithm incorporates certain concepts from their paper while utilizing both the existing offline data, known as historical data, and the sequentially revealed online data to estimate the arrival rate. We demonstrate that our algorithm achieves a superior ratio compared to theirs, even in the absence of any historical data, i.e., no prior knowledge of the arrival rate.

Recently, it has been an increasing trend to make use of historical data to aid in designing online algorithms (Azar et al. 2014, Correa et al. 2019, Kaplan et al. 2021, Zhang et al. 2022, Gorlezaei et al. 2022). Zhang et al. (2022) is the most related to ours. In their paper, they studied a maximum edge-weighted matching under the random order model, which is a restricted model of our online GAP setting. They leveraged the historical data in designing their matching algorithm and established the performance guarantee as a function of the size of historical data. Inspired by their algorithm, we develop our multi-phase algorithm in solving our multidimensional GAP problem. But note that we consider a Poisson arrival model while their work studied a random order model, and the multidimensional GAP model we study is more general than their online matching model. These differences necessitate a distinct and intricate analysis of the performance guarantee. In a concurrent study by Liu et al. (2023), historical data utilization is also explored within the context of the multidimensional GAP. They assume a random order model for the arrival process. To establish a performance guarantee for their algorithm, they make two additional assumptions: (1) dimension $D = 1$, (2) the ratio between the capacity of the offline bin and the demand of the online item is upper bounded by a constant. These assumptions impose more constraints on the problem setting compared to ours. In contrast, our research considers a general multi-dimensional GAP and adopts a Poisson arrival model with an unknown rate. This distinction significantly impacts both the algorithm design and the analysis, presenting unique challenges in our study.

2. Preliminaries

Let $[k]$ denote $\{1, \ldots, k\}$. We consider an online stochastic generalized assignment problem (GAP) as follows. We need to pack some online arriving items into some offline bins over a planning horizon of T periods. We use U to denote the set of bins. Each bin $u \in U$ is of D dimensions and has a capacity $\mathbf{C}_u = (C_u^1, \ldots, C_u^D) \in \mathbb{R}^D_{\geq 0}$. We use V to denote a set of item types, whose demand is of D dimensions. Packing an item whose type is $v \in V$ into a bin $u \in U$ consumes a capacity denoted by $\mathbf{r}_{uv} = (r_{uv}^1, \ldots, r_{uv}^D) \in \mathbb{R}^D_{\geq 0}$. We assume there are m item types in V. Packing an item of the type $v \in V$ into a bin $u \in U$ generates a non-negative reward of w_{uv}. Without loss of generality, we assume each item can be packed into any bin. If a pair of item and bin is incompatible, we can simply set its rewards w_{uv} to 0. Our goal is to maximize the total reward of the packing without violating the capacity constraints of all bins. We formulate this GAP problem in mathematical form as follows,

$$\max \sum_{u \in U, v \in V} w_{uv} x_{uv} \tag{1}$$

$$\text{s.t.} \sum_{u \in U} x_{uv} \leq n_v, \qquad \forall v \in V, \tag{1a}$$

$$\sum_{v \in V} r_{uv}^d x_{uv} \leq C_u^d, \qquad \forall u \in U, d \in [D], \tag{1b}$$

$$x_{uv} \in \{0, 1, \ldots, n_v\}, \qquad \forall u \in U, v \in V, \tag{1c}$$

where n_v denotes the number of online arriving items of type v and x_{uv} is the allocation decision, which represents the number of items of type v packed into the bin u.

We now define our online process. We are given a planning horizon of T and we assume T is large, which is a standard assumption in the online resource allocation literature (see Feldman et al. (2009), Huang and Shu (2021), Manshadi et al. (2012), Jaillet and Lu (2014)). We assume the online arrivals follow a type-specific Poisson process, i.e., the arrival rate of each item type v denoted by λ_v depends on its type. But $\lambda_v > 0$ is *unknown* for all $v \in V$. We assume each type's arrival process is independent from each other. Upon the arrival of one item, we need to make an immediate and irrevocable decision: pack it into one bin with enough space and generate a reward, or reject it.

Historical data. In this paper, we assume that the decision maker has some pre-existing offline data before the online process. We name them historical data in the subsequent presentation. We assume the historical data are generated from the *same* arrival model as in our online process in a time horizon of $h \cdot T$. In other words, the parameter $h \in [0,1]$ measures the size of the historical data. The average number of each item type v in the historical data is $hT\lambda_v$. The main focus of this paper is to scrutinize the performance of the proposed algorithm within a limited data setting. Consequently, we confine our considerations to the assumption $h \leq 1$. However, it's important to note that our algorithm and analysis can be extended to scenarios where $h > 1$, albeit with a more intricate competitive ratio.

Competitive ratio. We measure the performance of online algorithms by a competitive ratio. We define $\mathrm{ALG}(I)$ as the expected reward of the packing output by an online algorithm ALG on an instance I of our problem. The expectation is taken over random arrivals of online items during both the historical and online planning horizons and the randomized (if needed) algorithm. We then compare its performance with a clairvoyant optimal algorithm OPT, which holds the information of all the subsequent arrivals, i.e., each online arrival's type and its arriving time. We can similarly define $\mathrm{OPT}(I)$ as the expected reward of the packing output by OPT on I. For simplicity of the notion, we will call $\mathrm{OPT}(I)$ the offline optimal and drop I when there is no ambiguity, in the following analysis. Then the competitive ratio of ALG is defined as the minimum ratio of $\mathrm{ALG}(I)$ over $\mathrm{OPT}(I)$ among all instances I of our problem.

2.1. Important Lemmas

We next present two lemmas that are important in analyzing the performance of our algorithms. Due to page limit, we defer all the technical proofs in this paper to the appendix.

LEMMA 1 **(Number of Samples)**. *For a Poisson arrival process with an arrival rate $\lambda > 0$ during the time horizon of T, and denote $N = \lambda \cdot T$, we have that with probability $1 - e^{-\frac{N}{8}}$, the number of arrivals n is at least*

$$n \geq \frac{1}{2}N.$$

LEMMA 2 **(Arrival Rate Estimation)**. *For a Poisson arrival process with an unknown arrival rate $\lambda > 0$ that begins at time $t = 0$, if we observe n points arriving at time $t_1, t_2, \ldots, t_n$, we can estimate the arrival rate $\hat{\lambda} = \frac{t_n}{n}$, and for $0 < \delta < 1$, with probability $1 - \delta$ that*

$$(1 - \Delta) \cdot \hat{\lambda} \leq \lambda \leq (1 + \Delta) \cdot \hat{\lambda}$$

where $\Delta = \sqrt{\frac{4 \ln \frac{1}{\delta}}{n}}$.

3. Online Multidimensional GAP

In this section, we consider the online multidimensional GAP model. Inspired by Naori and Raz (2019), we categorize the edges connecting offline bins and online items into two types: heavy edges and light edges, according to the demand vector of the online items, and design a multi-phase algorithm (see Algorithm 1). Specifically, we define an edge (u, v) as a *light* edge if $r_{uv}^d \leq \frac{1}{2} C_u^d$ holds for all $d \in [D]$, otherwise, the edge is a *heavy* edge. Intuitively, for a heavy edge (u, v), we cannot pack more than one item of type v into the bin u. We use $\mathcal{E}^H$ and $\mathcal{E}^L$ to represent the set containing heavy edges and light edges, respectively.

We first reserve some periods in the beginning of the planning horizon as the sampling phase to collect arrival data. In other words, we simply reject all arrivals in the sampling phase to collect data to estimate the arrival rates. After that, we start our exploitation phases. In order to prevent lighter items from obstructing the packing of heavier items, we first deal with heavy edges to reach a better utilization of resources. Moreover, we apply a hybrid of a deterministic matching algorithm and an LP-based randomization algorithm to handle each type of edge. This hybrid approach enables efficient exploration of the solution space while minimizing the risk of getting trapped in local optima. This combination of deterministic and randomization algorithms allows us to strike a balance between efficient allocation search and adaptability to uncertain arrivals.

Remark: The algorithms exhibits a trade-off of the exploration (collecting data to learn arrival rate) and exploitation (allocate resources to maximize expected reward). A lengthier sampling period (larger α) intensifies the emphasis on exploration. This algorithm shares parallels with strategies used for solving multi-arm bandit (MAB) problems. However, it is crucial to assert that MAB algorithms are not directly transferrable to our scenario for the following reasons. First, our problem involves limited capacity for each resource, corresponding to options in MAB. This restricts the number of times each option can be chosen. Conversely, MAB permits unlimited plays for each option. Second, our goal is to comprehend the stochastic arrival process, unlike MAB where the focus is on random rewards. Lastly, our problem selects resources (options) based on rewards, while MAB plays options before knowing the reward.

We next explain our exposition phases in detail. In the second phase, we match heavy edges according to the optimal solution to a linear program $LP^H(\hat{\boldsymbol{\lambda}}, T')$ which is defined in (2), i.e., the resource u is matched to the request v with the probability that is proportional to $x^*_{uv}/(\hat{\lambda}_v T)$, where x^*_{uv} represents the optimal solution derived from $LP^H(\hat{\boldsymbol{\lambda}}, T')$. Here, $\hat{\boldsymbol{\lambda}}$ is the estimated arrival rate and $T' = (1-\alpha)T$ representing the actual time horizon where the offline resources can be consumed, i.e., excluding the sampling phase (see Steps 6-7 of Algorithm 1). We define $LP^H(\boldsymbol{\lambda}, T)$ as follows.

$$\max \sum_{u\in U, v\in V, (u,v)\in\mathcal{E}^H} w_{uv} x_{uv} \tag{2}$$

$$\text{s.t.} \sum_{u\in U,(u,v)\in\mathcal{E}^H} x_{uv} \le \lambda_v \cdot T, \qquad \forall v\in V, \tag{2a}$$

$$\sum_{v\in V,(u,v)\in\mathcal{E}^H} x_{uv} \le D, \qquad \forall u\in U, \tag{2b}$$

$$x_{uv}\in[0,1], \qquad \forall u\in U, v\in V, (u,v)\in\mathcal{E}^H. \tag{2c}$$

Here, x_{uv} is the decision variable representing the expected number of matches between bin u and item type v. Constraints (2a) bound the number of matches for each online type by its expected number of arrivals. Constraints (2b) bound the number of matches for each offline bin by the dimension D. Constraints (2b) hold because packing an item into a bin u in an heavy edge must break the conditions $r^d_{uv} \le \frac{1}{2}C^d_u, d\in[D]$ for at least one $d\in[D]$. As a result, for each $d \in [D]$, at most one item in a heavy edge can be packed into the bin u. Therefore, at most D items in total can be packed into each bin if only considering heavy edges. We next show that this LP provides an upper bound of the offline optimal value for the instances that only match heavy edges.

LEMMA 3. *The optimal value of $LP^H(\boldsymbol{\lambda}, T)$ provides an upper bound of the offline optimal for the instances that only heavy edges can be matched under known Poisson arrival model, where T is the online time horizon and $\boldsymbol{\lambda}$ represents the arrival rates.*

In the third phase, we solve a deterministic maximum matching problem in a bipartite graph. This graph consists of offline vertices on one side and the online arrivals within a time horizon of at most T on the other side. Note that we limit our consideration to up to T arrivals in order to maintain comparability with the offline optimal solution. Specifically, if the time horizon spanned by total arrivals, which includes both historical data and online arrivals, is less than or equal to T, we incorporate all the previous arrivals in constructing the bipartite graph. However, in cases where the time horizon exceeds T, we consider the online arrivals up until time $(1-h)T$ in combination with all the historical data that span a duration of hT without sacrificing generality. We then apply the maximum bipartite matching algorithm to select an offline resource for the arriving vertex. A match is made if the selected resource possesses adequate capacity to accommodate the vertex.

The fourth phase is the light LP phase, which is designed to solely deal with light edges. In this phase, we employ an LP-based randomization method to select resources for the items in light edges. Specifically, we consider a linear program $LP_0^L(\boldsymbol{\lambda}, T, C)$ defined in (3). Here, we specify the key parameters $\boldsymbol{\lambda}, T, C$ in the LP model to emphasize its dependence on those parameters, where C represents a matrix whose entries record C_u^d for all $u \in U$ and $d \in D$. In this phase, we only consider the unused capacity of the offline resource. In other words, we input the unused capacity for bin u in the dth dimension, denoted as $\bar{C}$, into the LP at Step 23 of Algorithm 1. After updating the estimate of $\hat{\boldsymbol{\lambda}}$ at Step 22, we utilize the solution to $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C})$ to instruct our allocation. We use $T' = (1-\eta)T$ to represent the total time horizon in the remaining online process.

$$\mathbf{max} \sum_{u \in U, v \in V, (u,v) \in \mathcal{E}^L} w_{uv} y_{uv} \tag{3}$$

$$\mathbf{s.t.} \sum_{u \in U, (u,v) \in \mathcal{E}^L} y_{uv} \le \lambda_v T, \qquad \forall v \in V, \tag{3a}$$

$$\sum_{v\in V,(u,v)\in\mathcal{E}^L} r_{uv}^d y_{uv} \le C_u^d, \qquad \forall u\in U, d\in[D], \tag{3b}$$

$$y_{uv}\ge 0, \qquad \forall u\in U, v\in V, (u,v)\in\mathcal{E}^L. \tag{3c}$$

LEMMA 4. *The optimal value of $LP_0^L(\boldsymbol{\lambda},T,C)$ is an upper bound of the offline optimal of the instance that only light edges can be matched under known Poisson arrival model where T is the online time horizon, C is the total capacity of offline bins and $\boldsymbol{\lambda}$ represents the parameters of the type-specific arrivals.*

In a similar fashion to the two phases dedicated to heavy edges, we include a light maximum packing phase that utilizes a deterministic matching algorithm in conjunction with a randomization algorithm used in the fourth phase. Note that multiple items can be packed in a bin for light edges, we deploy a linear program to determine the matching decision. Specifically,we denote V' as the set of all arrivals before $\min\{t,(1-h)T\}$, together with all the historical data and the present item of type v. The matching decision is made based on the optimal solution $\boldsymbol{y}$ to $LP_1^L(V')$ defined in (4), where the decision variables are $\{y_{uv}\}$.

$$\mathbf{max} \sum_{u\in U,v\in V',(u,v)\in\mathcal{E}^L} w_{uv}y_{uv} \tag{4}$$

$$\mathbf{s.t.} \sum_{u\in U,(u,v)\in\mathcal{E}^L} y_{uv}\le 1, \qquad \forall v\in V', \tag{4a}$$

$$\sum_{v\in V',(u,v)\in\mathcal{E}^L} r_{uv}^d y_{uv} \le C_u^d, \qquad \forall u\in U, d\in[D], \tag{4b}$$

$$y_{uv}\in[0,1], \qquad \forall u\in U, v\in V', (u,v)\in\mathcal{E}^L. \tag{4c}$$

Comparing this LP to LP (1), we notice that this LP provides an upper bound on the instance, considering only the matching of light edges since we only relax the integral constraints of decision variables. It is worthwhile to note that the capacity C here represents the total capacity, in contrast to the unused capacity described in LP (3).

In summary, the main idea of designing the exposition phases is as follows. We handle heavy edges first, followed by light edges. For each type of edge, we employ a hybrid algorithm that combines an LP-based randomization technique with a deterministic maximum matching technique to determine the matching decisions. Note that Phases 2 and 4 both involve the LP-based randomization technique, which is to determine the matching probability based on a proportion of the optimal solution to the LP, which is known as a scaling parameter. We use different scaling parameters in different phases. Specifically, we use γ to denote the scaling parameter for the phase handling heavy edges (Phase 2) and γ' to denote the parameter for the phases for light edges (Phase 4).

We next proceed to analyze the performance guarantee of this algorithm. Before we start the analysis for each phase, we first present two necessary concepts: OPT^H and OPT^L. For an instance of our problem, we use OPT^H to represent the expected reward of the offline optimal if only heavy edges can be matched, and OPT^L to denote the expected reward of the offline optimal if we can only match light edges. Then, the following lemma holds, since we can effectively decompose the offline optimal solution for the original instance into two separate solutions, where each solution consists exclusively of either heavy or light edges.

LEMMA 5. $OPT^H + OPT^L \geq OPT.$

3.1. Heavy LP Phase

First, we discuss the heavy LP phase. By bounding the gap between the optimal value of $LP^H(\hat{\boldsymbol{\lambda}}, T')$ and OPT^H and the probability of the event E that an offline bin u contains no item before time t where $t \in [\alpha \cdot T, \beta \cdot T)$, we can get the following lemmas. Here, we first define $\Delta = \sqrt{\frac{8ln\frac{1}{\delta}}{(h+\alpha)N}}$ and $N = \min_{v\in V} \lambda_v \cdot T$ and $0 < \delta < 1$, which is the corresponding error parameter specified in Lemma 2. By applying Lemmas 1 and 2, $(1-\Delta)\cdot\hat{\lambda}_v \leq \lambda_v \leq (1+\Delta)\cdot\hat{\lambda}_v$ holds for all $v \in V$ with some probability. Thus, we assume $(1-\Delta)\cdot\hat{\lambda}_v \leq \lambda_v \leq (1+\Delta)\cdot\hat{\lambda}_v$ in the subsequent analysis, if there is no further specification.

LEMMA 6. *The probability that an offline bin u contains no item at the end of the heavy LP phase is at least $e^{-\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}D}$. We denote it by q_α^β.*

LEMMA 7. *The expected reward during the heavy LP phase is weakly larger than $f_\alpha^\beta \cdot OPT^H$ where $f_\alpha^\beta = \frac{1}{D}(1-3\Delta)(1-\alpha)(1-q_\alpha^\beta)$.*

3.2. Heavy Maximum Matching Phase

We next analyze the heavy maximum matching phase. To determine the expected reward during this phase, we begin by comparing the expected weight of each matching edge with OPT^H. Then, for each possible value of t and the choices of β and η, we calculate the probability of the event E, which represents the scenario where bin u remains unpacked until time t. This information allows us to derive the expected reward for this phase.

Algorithm 1 Sample-based Algorithm for Online Multidimensional GAP

Input: Online arrivals of agents, history arrivals $h \cdot T$
Output: A feasible matching between online and offline vertices
Parameter: Phase parameters α, β, η, θ,and scaling parameters γ and γ' satisfying $0 \leq \alpha \leq \beta \leq \eta \leq \theta \leq 1$ and $0 \leq \gamma, \gamma' \leq 1$

while t increases from 0 to T continuously **do**
  if $0 \leq t < \alpha \cdot T$ **then** ▷ Sampling phase
    reject all online arrivals
  end if
  if $t = \alpha \cdot T$ **then** ▷ Estimation
    according to the arrival history $[-h \cdot T, \alpha \cdot T)$ and Lemma 2, estimate $\hat{\boldsymbol{\lambda}}$
    solve $LP^H(\hat{\boldsymbol{\lambda}}, T')$ where $T' = (1-\alpha)T$, and get the solution $\hat{\boldsymbol{x}}$
  end if
  if $\alpha \cdot T \leq t < \beta \cdot T$ **then** ▷ *Heavy* LP phase
    for each arrival i whose type is $v \in V$ **do**
      sample an offline vertex u with probability $p_{uv} = \gamma \frac{\hat{x}_{uv}}{\hat{\lambda}_v T'}$, match u and i if u is available
    end for
  end if
  if $\beta \cdot T \leq t < \eta \cdot T$ **then** ▷ *Heavy* maximum matching phase
    for each arrival i whose type is $v \in V$ **do**
      $V' = V([-h \cdot T, \min\{t, (1-h)T\})) \cup \{v\}$
      find optimal matching M' of $G' = (U, V', \mathcal{E}^H)$
      match i and u if $(u,v) \in M'$ and u is available
    end for
  end if
  if $t = \eta \cdot T$ **then** ▷ Estimation
    according to the arrival history during $[-h \cdot T, \eta \cdot T)$ and Lemma 2, estimate $\hat{\boldsymbol{\lambda}}$
    use $\bar{C}$ to record the unused capacity of each bin at time t
    solve $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C})$ where $T' = (1-\eta)T$ and get the solution $\hat{\boldsymbol{y}}$
  end if
  if $\eta \cdot T \leq t < \theta \cdot T$ **then** ▷ *Light* LP phase
    for each arrival i whose type is $v \in V$ **do**
      sample an offline vertex u with probability $p_{uv} = \gamma' \frac{\hat{y}_{uv}}{\hat{\lambda}_v T'}$, match u and i if u is available
    end for
  end if
  if $\theta \cdot T \leq t < T$ **then** ▷ *Light* maximum packing phase
    for each arrival i whose type is $v \in V$ **do**
      $V' = V([-h \cdot T, \min\{t, (1-h)T\})) \cup \{v\}$, solve $LP_1^L(V')$ and get the solution $\boldsymbol{y}$
      sample an offline vertex u with probability $p_{uv} = y_{uv}$, match u and i if u is available
    end for
  end if
end while

LEMMA 8. *The expected reward during the heavy maximum matching phase is at least* $q_\alpha^\beta f_\beta^\eta \cdot OPT^H$*, where* f_β^η *is defined as:*

$$f_\beta^\eta = \begin{cases} \frac{1}{D}(h+\beta)\ln\frac{h+\eta}{h+\beta}, & \eta \le 1-h \\ \frac{1}{D}(h+\beta)(\ln\frac{1}{h+\beta} + 1 - e^{1-h-\eta}), & \eta > 1-h \ge \beta \\ \frac{1}{D}(1-e^{-(\eta-\beta)}), & \beta > 1-h. \end{cases}$$

To facilitate subsequent analysis, we obtain a lower bound of the probability that u is not packed any item during the heavy maximum matching phase as below.

LEMMA 9. *Given the fact that bin* u *contains no items before the heavy maximum matching phase, the probability of the event* E *that bin* u *is not packed any item during the heavy maximum matching phase is at least*

$$q_\beta^\eta = \begin{cases} \frac{h+\beta}{h+\eta}, & \eta \le 1-h \\ (h+\beta)e^{1-h-\eta}, & \eta > 1-h \ge \beta \\ e^{-(\eta-\beta)}, & \beta > 1-h. \end{cases}$$

3.3. Light LP Phase

At time $t = \eta \cdot T$, we estimate the arrival rate again. Following Lemmas 1 and 2, we can update the estimation $\hat{\boldsymbol{\lambda}}$ that satisfies $(1-\Delta')\cdot\hat{\lambda}_v \le \lambda_v \le (1+\Delta')\cdot\hat{\lambda}_v$ where $\Delta' = \sqrt{\frac{8ln\frac{1}{\delta}}{(h+\eta)N}}$. We next establish a lower bound for the expected value of $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C})$ utilized in this phase. This estimate is derived by taking a portion of OPT^L and creating a feasible solution for $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C})$ using the optimal solution for $LP_0^L(\boldsymbol{\lambda}, T, C)$, whose value is an upper bound of OPT^L. By applying the union bound and Markov's inequality to calculate the probability of event E, which represents the availability of an offline bin u before time t after $\eta \cdot T$, given that no items were packed into bin u before $\eta \cdot T$, we can obtain a lower bound for the expected reward.

LEMMA 10. *The expected reward during the light LP phase is at least* $q_\alpha^\beta q_\beta^\eta f_\eta^\theta \cdot OPT^L$*, where* f_η^θ *is defined as below:*

$$f_\eta^\theta = (1-2\Delta')\gamma'(\theta-\eta)\left(1 - D\gamma'(1+\Delta')\frac{\theta-\eta}{1-\eta}\right).$$

Here, we also denote q_η^θ as an upper bound of the expected consumed fraction of some bin during the light LP phase, conditioning on the event that this bin has not been packed any item before the light LP phase.

LEMMA 11. *Conditioning on the event that bin u has not been packed any item before the light LP phase, the expected consumption of this bin in d-th dimension for each* $d \in [D]$ *is at most* $\gamma'(1+\Delta')\frac{\theta-\eta}{1-\eta}C_u^d$. *That is,* $q_\eta^\theta = \gamma'(1+\Delta')\frac{\theta-\eta}{1-\eta}$.

3.4. Light Maximum Packing Phase

According to the analysis of previous phase, the probability of the event E that a bin u has not been packed any items before time ηT is at least $q_\alpha^\beta q_\beta^\eta$.

Consider the arrival of an online vertex i with type $v \in V$ into the system at a specific time $t \in [\theta T, T)$, and let $\ell = (u,i)$ represent the corresponding edge chosen by Step 34 in Algorithm 1. By evaluating the lower bound of the expected weight of ℓ in comparison to OPT^L, as well as determining the probability of the event where the matching of the current edge is successful, we obtain the following lemma.

LEMMA 12. *The expected reward during the light maximum packing phase is at least* $q_\alpha^\beta q_\beta^\eta f_\theta^1 \cdot OPT^L$ *where* f_θ^1 *is*

$$f_\theta^1 = \begin{cases} (1+2D)(1-h-\theta) + 2D\ln(h+\theta) + h(1+2D\ln(h+\theta) - Dh) - 2D(1-\theta)q_\eta^\theta, & \theta \le 1-h \\ (1-\theta)(1-D(1-\theta)) - 2D(1-\theta)q_\eta^\theta, & \theta > 1-h. \end{cases}$$

3.5. Parametric Competitive Ratio Analysis

We summarize Lemmas 5, 7, 8, 10 and 12 to get the following theorem.

THEOREM 1. *Denote* $N = \min_{v\in V}\lambda_v \cdot T$. *For* $0<\delta<1$, *by choosing phase parameters* α, β, η *and* θ, *and scaling parameters* γ *and* γ' *satisfying* $0 \le \alpha \le \beta \le \eta \le \theta \le 1$ *and* $0 \le \gamma, \gamma' \le 1$, *with probability at least* $1 - 2m\delta - me^{\frac{-(h+\alpha)N}{8}} - me^{\frac{-(h+\eta)N}{8}}$, *Algorithm 1 has a competitive ratio of at least*

$$\max_{\alpha,\beta,\eta,\theta,\gamma,\gamma'} \min\{F^H, F^L\}$$

where

$$F^H = f_\alpha^\beta + \begin{cases} q_\alpha^\beta \cdot \frac{1}{D}(h+\beta)\ln\frac{h+\eta}{h+\beta}, & \eta \le 1-h \\ q_\alpha^\beta \cdot \frac{1}{D}(h+\beta)(\ln\frac{1}{h+\beta} + 1 - e^{1-h-\eta}), & \beta \le 1-h < \eta \\ q_\alpha^\beta \cdot \frac{1}{D}(1-e^{-(\eta-\beta)}), & 1-h < \beta, \end{cases}$$

$$F^L = \begin{cases} q_\alpha^\beta \cdot \frac{h+\beta}{h+\eta} \cdot \left(f_\eta^\theta + f_1\right), & \theta \le 1-h \\ q_\alpha^\beta \cdot \frac{h+\beta}{h+\eta} \cdot \left(f_\eta^\theta + f_2\right), & \eta \le 1-h < \theta \\ q_\alpha^\beta \cdot (h+\beta)e^{1-h-\eta} \cdot \left(f_\eta^\theta + f_2\right), & \beta \le 1-h < \eta \\ q_\alpha^\beta \cdot e^{-(\eta-\beta)} \cdot \left(f_\eta^\theta + f_2\right), & 1-h < \beta. \end{cases}$$

Here

$$\begin{cases} f_\alpha^\beta = \frac{1}{D}(1-3\Delta)(1-\alpha)(1-q_\alpha^\beta), \\ q_\alpha^\beta = e^{-\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}D}, \Delta = \sqrt{\frac{8ln\frac{1}{\delta}}{(h+\alpha)N}}, \end{cases}$$

$$\begin{cases} f_\eta^\theta = (1-2\Delta')\gamma'(\theta-\eta)(1-D\gamma'(1+\Delta')\frac{\theta-\eta}{1-\eta}), \\ q_\eta^\theta = \gamma'(1+\Delta')\frac{\theta-\eta}{1-\eta}, \Delta' = \sqrt{\frac{8ln\frac{1}{\delta}}{(h+\eta)N}}, \end{cases}$$

$$\begin{cases} f_1 = (1+2D)(1-h-\theta) + 2D\ln(h+\theta) + h(1+2D\ln(h+\theta) - Dh) - 2D(1-\theta)q_\eta^\theta, \\ f_2 = (1-\theta)(1-D(1-\theta)) - 2D(1-\theta)q_\eta^\theta. \end{cases}$$

4. Analysis for Special Cases

Note that the parametric competitive ratio in Theorem 1 is too complicated to figure out the optimal or near optimal choice of parameters. In this section, we will restrict our attention to some special cases to fine tune the parameters and then analyze the corresponding competitive ratios.

4.1. No historical data

In the case without any historical data, we can get the following result which improves the competitive ratio $\frac{e^{-0.25}}{4D+2}$ in Naori and Raz (2019).

PROPOSITION 1. *Denote $N = \min_{v\in V}\lambda_v \cdot T$. When $h = 0$ and N is large, for $0<\delta<1$, by setting $\alpha = C_0 N^{-\frac{1}{3}}, \beta = 0.935\frac{2D}{2D+1}, \eta = \theta = \frac{2D}{2D+1}, \gamma = 0.084\frac{2D+1}{D^2}$, where C_0 is a constant such that α satisfies $3\sqrt{\frac{8ln\frac{1}{\delta}}{\alpha N}} = \alpha$, with probability at least $1 - m\delta - me^{\frac{-(h+\alpha)N}{8}}$, the competitive ratio of Algorithm 1 is at least*

$$(1 - C_3 N^{-\frac{1}{3}}) \cdot \frac{e^{-0.225}}{4D+2},$$

where C_3 is a specific constant defined in the proof.

According to Proposition 1, there exist a threshold N_0 that satisfies $(1 - C_3 N_0^{-\frac{1}{3}})e^{-0.225} = e^{-0.25}$. When $N > N_0$, we have $(1 - C_3 N^{-\frac{1}{3}}) \cdot \frac{e^{-0.225}}{4D+2} > \frac{e^{-0.25}}{4D+2}$, i.e., we can improve the competitive ratio achieved in Naori and Raz (2019).

4.2. Heuristics by skipping certain phases

In this section, we fix some phase parameters to tune others and give some feasible lower bounds. For ease of presentation, we denote the heavy LP phase and light LP phase as LP phases and denote the heavy maximum matching phase and light maximum packing phase as max phases.

4.2.1. No Max Phases We first consider the special case that there are no max phases, i.e., $\beta = \eta$ and $\theta = 1$. In the following proposition, we fine tune the other parameters to obtain a feasible lower bound of competitive ratio.

PROPOSITION 2. *Denote $N = \min_{v \in V} \lambda_v \cdot T$. For $0 < \delta < 1$, by setting $\alpha = C_0 N^{-\frac{1}{3}}, \beta = \eta = \eta_1$, $\theta = 1$, $\gamma = 1$ and $\gamma' = \frac{1}{2D}$, where C_0 is a constant such that α satisfies $3\sqrt{\frac{8ln\frac{1}{\delta}}{\alpha N}} = \alpha$ and η_1 is the solution of $e^{D\eta} = \frac{5-\eta}{4}$, with a probability of at least $1 - 2m\delta - me^{\frac{-(h+\alpha)N}{8}} - me^{\frac{-(h+\eta)N}{8}}$, the competitive ratio of Algorithm 1 is at least*

$$\frac{1}{D} \cdot \frac{1-\eta_1}{5-\eta_1} \cdot (1 - C_3 N^{-\frac{1}{2}} (h + C_0 N^{-\frac{1}{3}})^{-\frac{1}{2}}),$$

where C_3 is a specific constant defined in the proof.

4.2.2. No LP phases In this section, we consider the following special choice of parameters: $\alpha = \beta$ and $\eta = \theta$, i.e., no heavy and light LP phases. From the above analysis, when there are no LP phases, the formula of competitive ratio becomes $\min\{F^H, F^L\}$ where

$$F^H = \begin{cases} \frac{1}{D}(h+\alpha)\ln\frac{h+\eta}{h+\alpha}, & \alpha \le \eta \le 1-h \\ \frac{1}{D}(h+\alpha)(\ln\frac{1}{h+\alpha} + 1 - e^{1-h-\eta}), & \alpha \le 1-h < \eta \end{cases}$$

$$F^L = \begin{cases} \frac{h+\alpha}{h+\eta} \cdot f_1, & \eta \le 1-h \\ (h+\alpha)e^{1-h-\eta} \cdot f_2, & \alpha \le 1-h < \eta. \end{cases}$$

Here, the values of f_1 and f_2 are updated as below:

$$f_\eta^1 = \begin{cases} f_1 = (1-(h+\eta))(1+2D) + 2D\ln(h+\eta) + h(1+2D\ln(h+\eta) - Dh), & \eta \le 1-h \\ f_2 = (1-\eta)(1-D(1-\eta)), & \eta > 1-h. \end{cases}$$

We further assume our choice of α is at most $1-h$. Based on this assumption, we can establish distinct lower bounds for the competitive ratio by fine-tuning the parameters at various values of h. Note that the derived competitive ratio here holds with a probability of one according to the proof of Theorem 1 when there is no LP phase.

PROPOSITION 3. *Denote h_0 as $(2D^2+1)(\sqrt{1+\frac{1}{4D^2}}-1)+\frac{1}{2D}$. We can achieve the corresponding competitive ratios under different choices of h. Specifically,*

- *if $h \le \frac{1}{2D}$, by setting $\alpha=\beta=\frac{2D}{2D+1}(1+h)e^{-\frac{f_1(2D+1)}{2(1+h)}}-h$, $\eta=\theta=\frac{2D}{2D+1}(1+h)-h$, we achieve a competitive ratio of $f_1 e^{-\frac{f_1(2D+1)}{2(1+h)}}$. Here, $f_1 = 1-2Dh+2D(1+h)\ln\left[\frac{2D(1+h)}{2D+1}\right]+h-Dh^2$;*
- *if $h \ge h_0$, by setting $\alpha=\beta=1-h$, $\eta=\theta=2-\frac{1}{2D}-\sqrt{1+\frac{1}{4D^2}}$, we achieve a competitive ratio of $e^{1-h-\eta}(1-\eta)(1-D(1-\eta))$;*
- *if $\frac{1}{2D}<h<h_0$, by setting $\theta=\eta=1-\frac{1}{2D}$, $\beta=\alpha$ and*

$$\alpha=\begin{cases}\max\{\alpha_1,\alpha_2,0\}, & \alpha_1\le 1-h\\ 1-h, & \alpha_1 \ge 1-h,\end{cases}$$

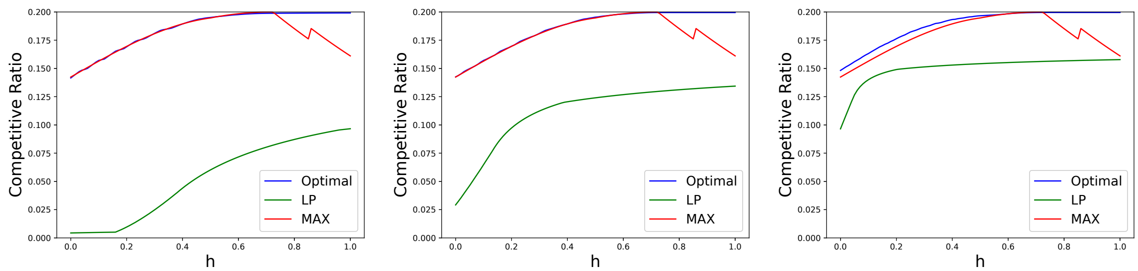
where $\alpha_1=\exp\{1-\frac{5}{4}e^{\frac{1}{2D}-h}\}-h$ and $\alpha_2=\exp\{-e^{\frac{1}{2D}-h}\}-h$, we achieve a competitive ratio of

$$\begin{cases}\frac{1}{D}(h+\alpha)(\ln\frac{1}{h+\alpha}+1-e^{\frac{1}{2D}-h}), & \alpha_1\le 1-h\\ e^{\frac{1}{2D}-h}\frac{1}{4D}, & \alpha_1\ge 1-h.\end{cases}$$

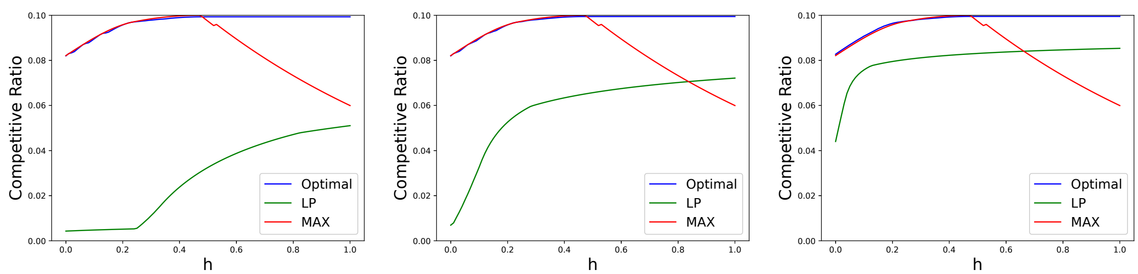
We next compare the competitive ratios of the heuristics with Algorithm 1 under the optimal choices of the four phases parameters and the two scaling parameters in Figure 1. It is worth noting that the analysis for the heuristics with no max phases, or equivalently only LP phases in Section 4.2.1 contains some approximation under the assumption that N is large, which may lead to a very poor performance when N is small. As a result, we enumerate an optimal α and use the rest parameters suggested in Proposition 2 for our analysis. For the heuristics with no LP phases, we directly apply the parameters suggested by Proposition 3.

Figure 1 compares the competitive ratio across different values of D, N and h, from which we find the heuristics with no LP phases, or equivalently those with only max phases perform very close to Algorithm 1. The competitive ratio of the heuristics with only max phases decreases when h surpasses a certain threshold denoted as $h_0 := (2D^2+1)(\sqrt{1+\frac{1}{4D^2}}-1)+\frac{1}{2D}$. This threshold is determined by the second case in Proposition 3. When h gets larger, the competitive ratio of the heuristics with only LP phases increases. Furthermore, in most cases except the case where $D=1$ and $N=2000$, the competitive ratio for the heuristics with only LP phases can reach a comparable or even higher competitive ratio than those with only max phases when h is close to 1.

In summary, we provide a guideline on fine tuning the parameters when different sizes of historical data are available, i.e., h takes different values. Specifically, when h is small, we can adopt the heuristics by skipping LP phases (see Section 4.2.2) and when h is large, we can use the heuristics by skipping max phases (see Section 4.2.1), which allows us to reach a relatively good performance.



(a) $D=1,\ N=2000$ (b) $D=1,\ N=10000$ (c) $D=1,\ N=100000$



(d) $D=2,\ N=2000$ (e) $D=2,\ N=10000$ (f) $D=2,\ N=100000$

Figure 1 Comparisons of competitive ratios under three parameter settings

4.3. Online matching problems

In this section, we restrict to a simplified setting where the dimension $D=1$ and both the capacity of offline bins and the demand of online items are set to 1. Under this setting, the problem reduces to the traditional online edge-weighted bipartite matching, where one side is offline vertices and the other side is online vertices. Since there is no light edge in this setting, we can simplify Theorem 1 by setting $\eta=\theta=1$ to get the following statement.

Corollary 1. *Denote $N=\min_{v\in V}\lambda_v\cdot T$. Under the online matching setting, for $0<\delta<1$, by choosing parameters α, β and γ satisfying $0\le\alpha\le\beta\le 1$ and $0\le\gamma\le 1$, with a probability of at least $1-m\delta-me^{\frac{-(h+\alpha)N}{8}}$, Algorithm 1 has a competitive ratio of at least:*

$$\begin{cases}(1-3\Delta)(1-\alpha)(1-e^{-\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}})+e^{-\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}}(h+\beta)(ln\frac{1}{h+\beta}+1-e^{-h}), & \beta\le 1-h\\ (1-3\Delta)(1-\alpha)(1-e^{-\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}})+e^{-\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}}(1-e^{-(1-\beta)}), & \beta>1-h,\end{cases}$$

where $\Delta = \sqrt{\frac{8ln\frac{1}{\delta}}{(h+\alpha)N}}$.

We first note that most of our ratios are between $\frac{1}{e}$ and $1-\frac{1}{e}$, where the former is the state-of-art ratio for the random order model given from Kesselheim et al. (2013) and the latter is the ratio of the typical algorithm for *known* Poisson model proposed by Feldman et al. (2009). This implies that incorporating the information on arrival model (Poisson arrival) can help improve the algorithm's performance even when we have no information on its arrival rate, equivalently no historical data ($h=0$).

We compare the bound of competitive ratios at different h and N in Figure 2a. From the figure, we first observe that when h increases, our competitive ratio increases, which indicates that adding historical data indeed help improve our algorithm's performance. Second, we also find that the value of N also significantly affects the algorithm's performance. Note that the value of N indicate the quality of data measured by $\min_{v\in V}\lambda_v$ for a fixed time horizon T. The dependence on N indicates that when $\min_{v\in V}\lambda_v$ is large for a given planning horizon, i.e., no item type is underrepresented, our algorithm performs well. In other words, we observe both quality and quantify of data will affect our algorithm's performance.

We then discuss the effect of historical data's quantity (h) and quality (N) on the tradeoff between exploration and exploitation measured by the values of α and β. We plot the optimal α and β at different h and N in Figures 2b and 2c, respectively. After fixing h, we plot the portion of three phases under different N in Figures 2d, 2e and 2f. Similarly, after fixing N, we also plot the portion of three phases under different choices of h in Figures 2g, 2h and 2i. In these six figures, the blue, green and red regions correspond to sampling phase, (heavy) LP phase and maximum matching phase, respectively.

To see the effect on the optimal choice of α, from Figure 2b, we observe that when either h or N increases, the optimal α decreases. It implies that when the historical data size increases, we can shorten the exploration period (sampling phase before the time point αT) and exploit the decision earlier to get better performance. On the other hand, if the data quality is high, i.e., each item type has sufficient arrivals for us to learn its arrival rate, we can also shorten the exploration period. As shown in Figures 2e, 2f, 2h and 2i, when h and N are large enough, i.e., a great amount of historical data, we can even ignore the sampling phase and initiate the exploitation directly.

We then analyze the effect on the choice of the hyperparameter β. From Figure 2c, we see that a larger β is needed when N increases or h decreases. Note that when N increases, the optimal α decreases according to our earlier analysis. This concludes that more periods are reserved for the LP phase with fewer periods for maximum matching phase when N increases. Such a trend can also be seen in Figures 2d, 2e and 2f. It implies that increasing data quality ($\min_{v\in V}\lambda_v$) can result in a more precise estimate of arrival rates and provide more substantial improvements in the LP phase compared to that in the maximum matching phase. Conversely, our analysis indicates that the performance of the maximum matching phase is primarily determined by the ratio of the historical data to the online data, as measured by h. Thus, an increase in h can improve the performance of the maximum matching phase more than that of the LP phase, which implies β decreases with h.

Further, under the restricted online matching setting, if there is no historical sample, we can set $\beta = 1$ and choose one α such that $1 - 3\Delta = 1 - \alpha$. In this case, we can achieve a relatively good performance guarantee.

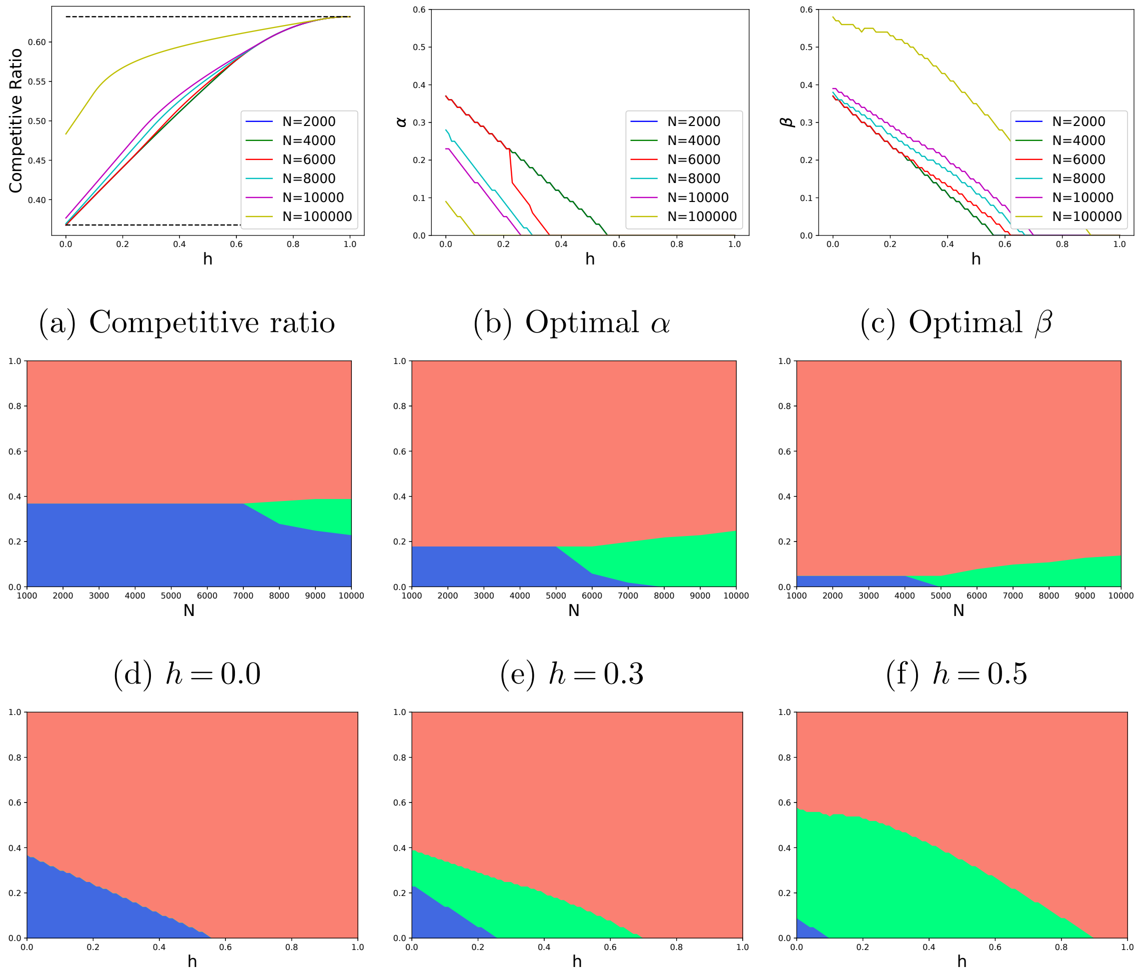
Corollary 2. *Denote $N = \min_{v\in V}\lambda_v \cdot T$. Under the online matching setting, without historical samples ($h = 0$), for $0 < \delta < 1$, with a probability of at least $1 - m\delta - me^{\frac{-\alpha N}{8}}$, Algorithm 1 can achieve a competitive ratio with at least $\left(1 - 4\sqrt[3]{\frac{9ln\frac{1}{\delta}}{N}}\right)(1-\frac{1}{e})$ by choosing $\alpha = \sqrt[3]{\frac{72ln\frac{1}{\delta}}{N}}$ and $\beta = \gamma = 1$.*

5. Experiments

In this section, we first utilize a dataset obtained from EverySender, a well-known task assignment platform, as described in the study by Tong et al. (2016) to examine the effectiveness of our algorithms in solving the online matching problem. Following that, we proceed to evaluate the performance of our algorithms on a synthetic dataset, specifically focusing on the online multidimensional GAP problem.

5.1. Online Matching

Dataset and preprocessing. EverySender dataset includes a set of workers and tasks. Each worker and task has a location (x, y). The data also provide the successful rate for each worker and payoff for each task. We process the data in the following way. We treat each worker as an online item and each task as an offline bin. We divide the map into a grid map



(a) Competitive ratio (b) Optimal α (c) Optimal β

(d) $h = 0.0$ (e) $h = 0.3$ (f) $h = 0.5$

(g) $N = 2000$ (h) $N = 10000$ (i) $N = 100000$

Figure 2 Illustrations of Proposition 1

where each grid has a size of (dx, dy), and group every worker/task in the same grid as one type. As a result, we generate a normalized location $(\tilde{x} = \lfloor x/dx \rfloor, \tilde{y} = \lfloor y/dy \rfloor)$ for each type of worker and task. We use $(\tilde{x}, \tilde{y})$ to indicate the worker/task's type. We use the frequency of each worker type to approximate its arrival rate. We use the average successful rate (payoff) of the same type of workers (tasks) as the successful rate (payoff) of this type. For each pair of worker and task, we add an edge between them if the Euclidean distance between them is smaller than a given threshold, and the weight of this edge is the product of the corresponding successful rate and payoff.

Algorithms. We test two heuristic algorithms based on Algorithm 1 and a greedy algorithm. Since we consider our algorithm in the online matching setting here, we set $\eta = \theta = 1$ directly.

- GRD: The greedy algorithm. For each arrival i of type v, match it to the available offline vertex u with the largest weight w_{uv}.
- SAM1: Algorithm 1 with $\alpha = \max\{e^{e^{-h}} - 1, 0\}$, $\beta = 1 - h$ and $\gamma = 1$.
- SAM2: Algorithm 1 with $\alpha = \max\{e^{e^{-h}} - 1, 0\}$, $\beta = 1$ and $\gamma = 1$.

According to Corollary 1, the optimal α and β depends on the values of h and N. However, N is not known by us in practice. We then choose a universal value of α and β to run our algorithm. Specifically, we set $\alpha = \max\{e^{e^{-h}} - 1, 0\}$ as suggested by Zhang et al. (2022), since this α maximizes the ratio in Corollary 1 when there is no LP phase. We choose two values for β: $\beta = 1 - h$ and $\beta = 1$. These two values are chosen because according to Corollary 1, the analyses for $\beta \leq 1 - h$ and $\beta > 1 - h$ are different, which implies $\beta = 1 - h$ is a critical value. $\beta = 1$ is considered since it refers to a special case where only LP phase is used in the exploitation period.

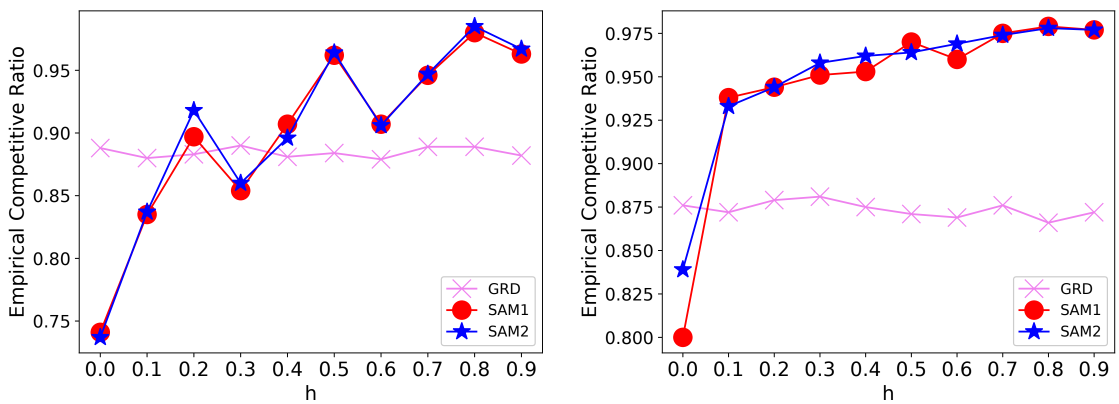
We test over $T = 1000, 2000, \cdots, 5000$ and $h = 0, 0.1, 0.2, \cdots, 0.9$. For each h and T, we generate an arrival sequence of a length of T and a history sequence of a length of $h \cdot T$ according to the arrival rates. We then run algorithms over the sequence to get the corresponding total reward. We repeat the procedure $M = 50$ times to get the average reward. To evaluate the competitive ratio, we first calculate the offline optimal reward in the following way. For each arrival sequence, we solve the maximum matching problem to get an optimal matching decision and evaluate its reward. We take average of these optimal rewards to get its offline average reward. We use the ratio between an algorithm's average reward and the offline average reward denoted by the *empirical competitive ratio* as the performance metric.

Results and discussion. We compare the ratio of different algorithms at different h and T and summarize the results in Figure 3 and Figure 4.

In Figure 3, we test different h, fixing $T = 1000$ and $T = 2000$. We can see that when h becomes larger, SAM1 and SAM2's performance shows an increasing trend and GRD's performance keeps almost unchanged. This is consistent with the analysis in Corollary 1 that when h goes larger, the performance of our algorithms becomes better. By comparing SAM1 and SAM2, we find that SAM2 does not always dominate SAM1 under different parameters (h and T). In fact, according to Figure 3b, when T is large, SAM1 consistently outperforms SAM2 when h is large. It implies that adding the maximum matching phase indeed helps improve the algorithm's performance in many instances.

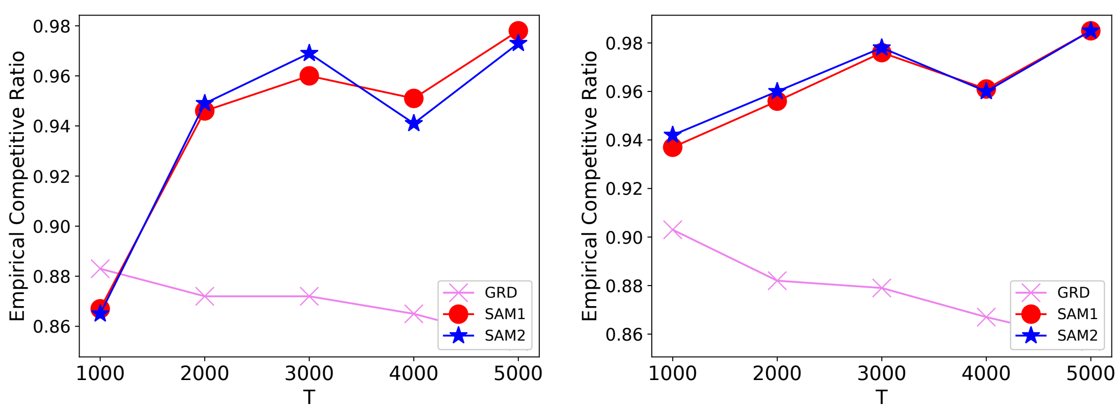
In Figure 4, we test different T when fixing $h = 0.3, 0.5$. We find that when T is larger, our algorithm's performance becomes better in general. This is again consistent with our findings in Corollary 1. We also find the gap between $T = 5000$ and $T = 1000$ becomes smaller when h increases. This is because if we have many historical data, we do not need a large planning horizon to achieve a good performance.

In summary, increasing the historical data size (a larger h) or increasing the planning horizon (a larger T) helps improve the performance of our heuristic algorithms even when N is small. Our heuristic algorithms can outperform greedy algorithm when h and T are large.



(a) $T = 1000$ (b) $T = 2000$

Figure 3 **Different** h



(a) $h = 0.3$ (b) $h = 0.5$

Figure 4 **Different** T

5.2. Online Multidimensional GAP

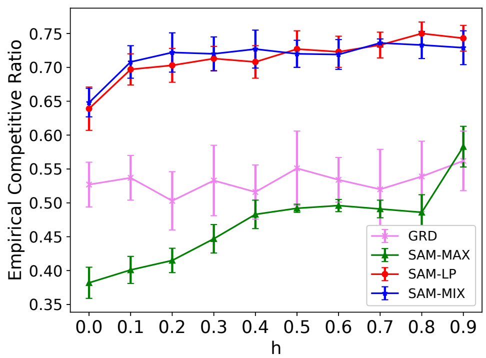
Dataset and preprocessing. Let $\mathcal{U}[a,b]$ denote a uniform distribution on the interval $[a,b]$. A problem instance is generated as follows. First we define $U=[m]$ and $V=[n]$. For each $u\in U$ and $v\in V$, we define an edge (u,v) with weight w_{uv} which is generated from a uniform distribution $\mathcal{U}[0,1]$. For the capacity of each bin $u\in U$, we set the capacity of each dimension as 1. For each edge (u,v), the d-th demand r_{uv}^d is generated from a uniform distribution $\mathcal{U}[0,1]$. For the arrival rates, we first generate a value l_v according to a uniform distribution $\mathcal{U}[0,1]$, then normalize the value to get the arrival rate for each v, i.e., $\lambda_v=\frac{l_v}{\sum_v l_v}$. After the normalization, the expected number of total arrivals is T.

Algorithms. We test three heuristic algorithms based on Algorithm 1 and a greedy algorithm.

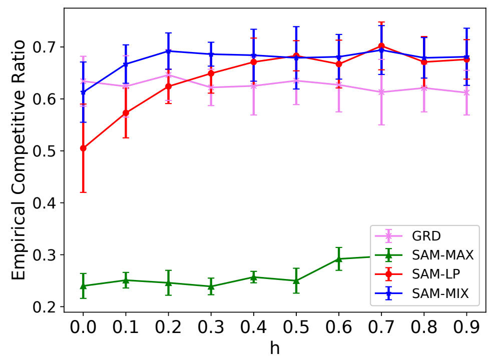
- GRD: A greedy algorithm. For each arrival i of type v, match it to the available offline vertex u with the largest edge weight w_{uv}. Here "available" means u has enough capacity for edge (u,v).
- SAM-MAX: The Algorithm 1 has sampling phase and max phases, the parameters are chosen according to Proposition 3.
- SAM-LP: The Algorithm 1 has sampling phase and LP phases. $\alpha=0.05$, $\beta=\eta$ where η is the solution of $e^{D\eta}=\frac{5-\eta}{4}$. $\theta=1$, $\gamma=\gamma'=1$. Specifically, $\eta\approx 0.185$ under $D=1$ and $\eta\approx 0.101$ under $D=2$.
- SAM-MIX: The Algorithm 1 has all phases. $\alpha=0.05$, $\beta=\alpha+\frac{\eta-\alpha}{2}$ where η is the solution of $e^{D\eta}=\frac{5-\eta}{4}$. $\theta=\eta+\frac{1-\eta}{2}$, $\gamma=\gamma'=1$.

REMARK 1. Note that for SAM-LP, we do not know the exact value of N, hence we simply choose a small value of $\alpha=0.05$. β, η, γ are chosen according to the Proposition 2. For γ', we choose $\gamma'=1$ because the value $\gamma=\frac{1}{2D}$ suggested by Proposition 2 is too conservative. The intuition of setting γ' to 1 is because when there is no other phase after the light LP phase, a larger value of γ' is more cost-effective. For SAM-MIX, we choose the same values of α, η, γ and γ' as in SAM-LP, and divide the heavy (or light) LP phase of SAM-LP into heavy (or light) LP phase and heavy (or light) max phase with equal size.

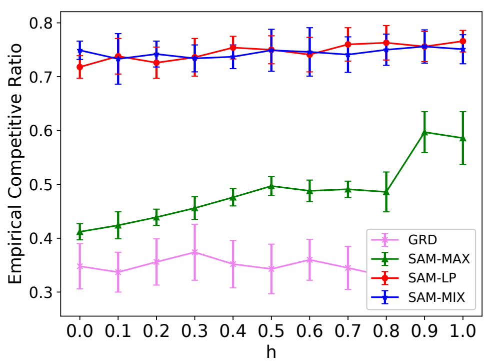
We use the parameters $D=1,2$, $m=n=10$ to generate our randomized graph G, and we generate 10 graphs for each D. For each graph, we test over $T=50,100,\cdots,500$ and $h=0,0.1,0.2,\cdots,0.9$. We then test the empirical competitive ratio following the same procedure as in the online matching problem above. We test over smaller Ts than those in the online matching problem because (1) the offline optimal packing problem here is NP-hard and time-consuming; and (2) these smaller Ts are already good enough for our sampling algorithms. For different graphs, we not only give the average empirical competitive ratio for all graphs, but also show the standard errors of the empirical competitive ratios by error bars to measure the robustness of different algorithms.



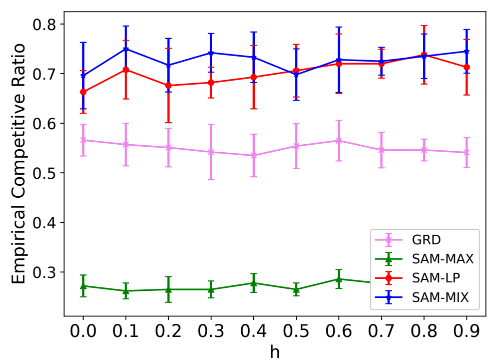
(a) $D = 1, T = 100$



(b) $D = 2, T = 100$



(c) $D = 1, T = 250$

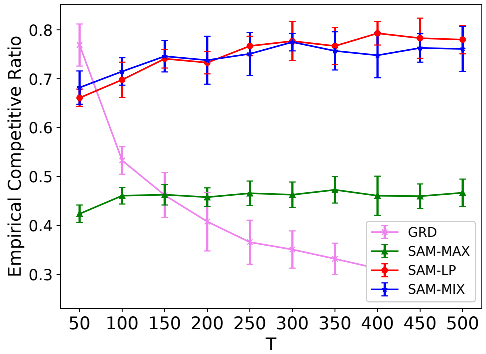


(d) $D = 2, T = 250$

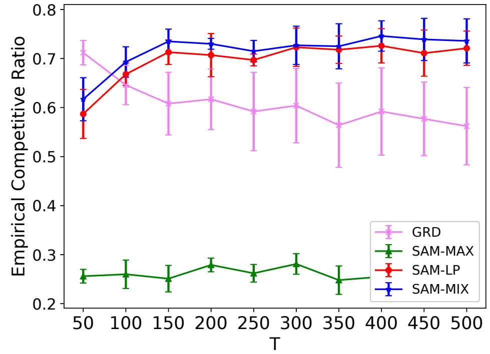
Figure 5 Different h

Results and discussion. We summarize the results in Figure 5 and Figure 6. Each error bar shows the average empirical competitive ratio ± standard error.

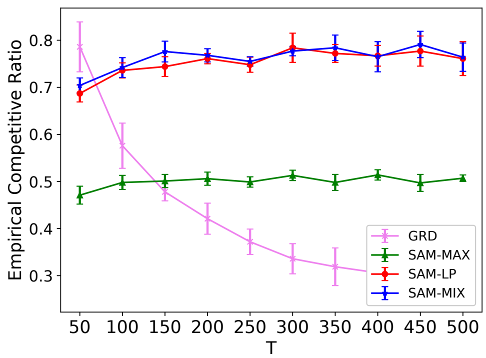
In Figure 5, we test different h, fixing $T = 100, 250$ and $D = 1, 2$. Generally speaking, either SAM-MIX or SAM-LP has the best performance, i.e., achieving the largest ratio in all cases except the case that $h = 0$, $T = 100$ and $D = 2$. This means in practice, we do not need a very large sample size to achieve a good performance for SAM-MIX or SAM-LP as discussed in Section 3. SAM-MAX's performance is much worse than the performance of SAM-MIX and SAM-LP, hence we focus on SAM-MIX and SAM-LP in the following analysis.



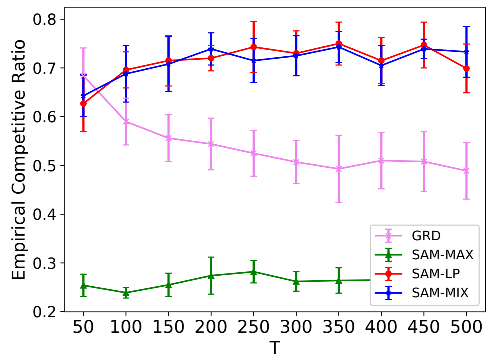
(a) $D=1, h=0.3$



(b) $D=2, h=0.3$



(c) $D=1, h=0.5$



(d) $D=2, h=0.5$

Figure 6 **Different** T

We find out that when h goes larger, the performance of Sam-Mix and Sam-LP increases. The increasing trend shows a diminishing decreasing pattern. This is because when we have enough data, the estimation is already accurate enough, then a larger historical data size h makes a smaller change.

We further find that the performances of Sam-Mix and Sam-LP are close in most cases, except the case when $h<0.5$ and $D=2$, where the gap between Sam-Mix and Sam-LP is relatively large. This is because when D is large, the problem becomes more complicated and needs more accurate estimation of the arrival rates to get a good solution of the LP. When h is small, the estimation of arrival rates is less inaccurate, which leads to a poor performance of LP phases in Sam-LP, whereas Sam-Mix incorporates the max phases that do not depend on the estimation to reach a better performance.

In Figure 6, we test different T when fixing $h=0.3, 0.5$ and $D=1,2$. In general, we find out that Sam-LP and Sam-Mix outperform other two baselines except when $T=50$. This is because when $T=50$, the number of samples are too small (the type of online vertices are $n = 10$), the ratios of our algorithms should be small. When T becomes larger, the performance of Sam-Mix and Sam-LP is increasing because we have more samples and the increasing speed is decreasing. After T is large enough ($T \geq 250$ when $h = 0.5$), the performance of Sam-Mix and Sam-LP becomes stable. When $h = 0.3$, $D = 2$ and $T < 250$, there is a significant gap between Sam-Mix and Sam-LP, and this can be explained by the similar analysis as we discuss for Figure 5.

To summarize, Sam-Mix has the best performance among all tested algorithms and Sam-LP can achieve similar performance as Sam-Mix when $h > 0.5$ and $T > 250$. When h and T is increasing, the performance of Sam-Mix and Sam-LP becomes better. The marginal utility of h and T is decreasing when h and T are large. The performance becomes stable when h and T are larger than some thresholds, i.e., if we already have "enough data" (large historical data h and planning horizon T), we do not need more data. In practice, the thresholds of h and T are small comparing with the theoretical analysis. We can use Sam-Mix for all non-trivial cases ($h > 0$ and $T > 50$), and if we have larger h, T or small D, we can also use Sam-LP which is more efficient than Sam-Mix.

6. Conclusions

We study the online multidimensional GAP in this paper. By providing a sample-based multi-phase algorithm, we provide a parametric performance guarantee. From the parametric form of the competitive ratio, we analyze the effect of historical data size and the dimension of capacity (demand) on the algorithm's performance. Finally, we test our algorithms by a real-world dataset (online matching) and a synthetic dataset (online multidimensional GAP).

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Appendix A: Missing proofs in Section 2

A.1. Proof of Lemma 1

Lemma 1 **(Number of Samples)**. *For a Poisson arrival process with an arrival rate $\lambda > 0$ during the time horizon of T, and denote $N = \lambda \cdot T$, we have that with probability $1 - e^{-\frac{N}{8}}$, the number of arrivals n is at least*

$$n \geq \frac{1}{2} N.$$

Proof The number of arrivals n follows a Poisson distribution with parameter N. According to Fact 4 from Canonne (2019), we can directly prove this lemma. □

A.2. Proof of Lemma 2

Lemma 2 **(Arrival Rate Estimation)**. *For a Poisson arrival process with an unknown arrival rate $\lambda >$ 0 that begins at time $t = 0$, if we observe n points arriving at time $t_1, t_2, \ldots, t_n$, we can estimate the arrival rate $\hat{\lambda} = \frac{t_n}{n}$, and for $0 < \delta < 1$, with probability $1 - \delta$ that*

$$(1 - \Delta) \cdot \hat{\lambda} \leq \lambda \leq (1 + \Delta) \cdot \hat{\lambda}$$

where $\Delta = \sqrt{\frac{4 \ln \frac{1}{\delta}}{n}}$.

Proof According to the property of Poisson arrival process, the differences between consecutive two t_is follow an exponential distribution. Chapter 2 of Wainwright (2019) gives us the concentration bounds of an exponential distribution random variable. □

Appendix B: Missing proofs in Section 3

B.1. Proof of Lemma 3

Lemma 3. *The optimal value of $LP^H(\boldsymbol{\lambda}, T)$ provides an upper bound of the offline optimal for the instances that only heavy edges can be matched under known Poisson arrival model, where T is the online time horizon and $\boldsymbol{\lambda}$ represents the arrival rates.*

Proof Denote r as a realization of instance I which is a possible input sequence of online item types. We then define $x^r_{u,v}$ as the number of packing of online item v into offline bin u by the offline optimal under realization r and P_r as the probability of the realization r. Here, only heavy edges can be considered in the offline optimal. We next denote x^*_{uv} as $\sum_r P_r x^r_{u,v}$, which represents the expected number of packing v into u by the offline optimal. Notice that $\mathrm{OPT}(I) = \sum_r P_r \sum_{u,v} x^r_{u,v} w_{uv} = \sum_{u,v} w_{uv} (\sum_r P_r x^r_{u,v})$, equal to $\sum_{u,v} w_{uv} x^*_{uv}$. It suffices to show $\{x^*_{uv}\}$ is a feasible solution to $LP^H(\boldsymbol{\lambda}, T)$.

For Constraints (2a), $\forall v \in V$, $\sum_{u \in U, (u,v) \in \mathcal{E}^H} x^*_{uv} = \sum_r P_r \sum_{u \in U, (u,v) \in \mathcal{E}^H} x^r_{u,v} \leq \sum_r P_r N^r_v = \lambda_v T$, where N^r_v denotes the number of items of type v in realization r. The inequality holds because the total number of packing of items of type v cannot larger than the number of arriving items of type v, while the last equality is from the linearity of expectation.

For Constraints (2b), $\forall u \in U$, we have $\sum_{v \in V, (u,v) \in \mathcal{E}^H} x^*_{uv} = \sum_r P_r \sum_{v \in V, (u,v) \in \mathcal{E}^H} x^r_{u,v} \leq \sum_r P_r D = D$. The reason for the inequality is as follows. Each $x^r_{u,v}$ is an integer which represents the number of packing an item of type v into bin u under realization r. Since only heavy edges are considered, each packing must break the conditions $r^d_{uv} \le \frac{1}{2}C^d_u, d \in [D]$ for at least a $d \in [D]$. That is, for each $d \in [D]$, at most one item that break the condition $r^d_{uv} \le \frac{1}{2}C^d_u$ can be packed into the bin u. Hence at most D items can be packed into one bin. The argument above that at most one item that break the condition $r^d_{uv} \le \frac{1}{2}C^d_u$ can be packed into one bin u for each $d \in [D]$ implies that each $x^r_{u,v}$ is at most 1. Thus, each $x^*_{u,v}$ is also at most 1, which satisfies Constraints (2c). □

B.2. Proof of Lemma 4

Lemma 4. *The optimal value of $LP^L_0(\boldsymbol{\lambda}, T, C)$ is an upper bound of the offline optimal of the instance that only light edges can be matched under known Poisson arrival model where T is the online time horizon, C is the total capacity of offline bins and $\boldsymbol{\lambda}$ represents the parameters of the type-specific arrivals.*

Proof We adopt a similar setup as in the proof of Lemma 3 and define the realization r, the $y^r_{u,v}$ and the y^*_{uv} in the same way. Here, the only difference is that we only consider light edges can be used in the offline optimal. Notice that $\mathrm{OPT}(I) = \sum_r P_r \sum_{u,v} y^r_{u,v} w_{uv} = \sum_{u,v} w_{uv}(\sum_r P_r y^r_{u,v})$, equal to $\sum_{u,v} w_{uv} y^*_{uv}$. It suffices to show $\{y^*_{uv}\}$ is a feasible solution to $LP^L_0(\boldsymbol{\lambda}, T, C)$.

For Constraints (3a), for each $v \in V$, $\sum_{u \in U, (u,v) \in \mathcal{E}^L} y^*_{uv} = \sum_r P_r \sum_{u \in U, (u,v) \in \mathcal{E}^L} y^r_{u,v} \le \sum_r P_r N^r_v = \lambda_v T$, where N^r_v denotes the number of item type v in realization r. The inequality holds because the total number of packing of items of type v cannot exceed the number of arriving items of type v, while the last equality is from the linearity of expectation.

For Constraints (3b), $\forall u \in U, d \in [D]$, we have $\sum_{v \in V, (u,v) \in \mathcal{E}^L} r^d_{uv} y^*_{uv} = \sum_r P_r \sum_{v \in V, (u,v) \in \mathcal{E}^L} r^d_{uv} y^r_{u,v} \le \sum_r P_r C^d_u = C^d_u$. The inequality holds because the capacity cannot be exceeded from the feasibility of the allocation under each realization r. □

B.3. Proof of Lemmas 6 and 7

To prove these two lemmas, we need to introduce another two lemmas first.

Lemma 13. *The optimal value of $LP^H(\hat{\boldsymbol{\lambda}}, T')$ is lower bounded by $\frac{1-\alpha}{1+\Delta}$ OPT^H.*

Proof From Lemma 3, the optimal value of $LP^H(\boldsymbol{\lambda}, T)$ is an upper bound of the offline optimal OPT^H in the original problem. It suffices to show the optimal value of $LP^H(\hat{\boldsymbol{\lambda}}, T')$ is lower bounded by the product of $\frac{1-\alpha}{1+\Delta}$ and the optimal value of $LP^H(\boldsymbol{\lambda}, T)$. We assume the optimal solution of $LP^H(\boldsymbol{\lambda}, T)$ is $\{x^*_{uv}\}$. We next show $\{\frac{1-\alpha}{1+\Delta} x^*_{uv}\}$ is a feasible solution of $LP^H(\hat{\boldsymbol{\lambda}}, T')$.

For Constraints (2a), for each $v \in V$, from the feasibility of $\{x^*_{uv}\}$ in $LP^H(\boldsymbol{\lambda}, T)$, we have $\sum_{u \in U} x^*_{uv} \le \lambda_v \cdot T$. This means $\sum_{u \in U} \frac{1-\alpha}{1+\Delta} x^*_{uv} \le \frac{1-\alpha}{1+\Delta} \lambda_v \cdot T = \frac{\lambda_v}{1+\Delta} \cdot (1-\alpha)T$. This is upper bounded by $\hat{\lambda}_v \cdot T'$ under our assumption, which corresponds to Constraints (2a) in $LP^H(\hat{\boldsymbol{\lambda}}, T')$. For Constraints (2b) and (2c), since $0 \le \frac{1-\alpha}{1+\Delta} \le 1$, we can directly induce the feasibility of these two constraints in $LP^H(\hat{\boldsymbol{\lambda}}, T')$.

Because $\{\frac{1-\alpha}{1+\Delta} x^*_{uv}\}$ is a feasible solution of $LP^H(\hat{\boldsymbol{\lambda}}, T')$ with the specified probability, the optimal value of $LP^H(\hat{\boldsymbol{\lambda}}, T')$ is weakly larger than the corresponding value of the feasible solution $\{\frac{1-\alpha}{1+\Delta} x^*_{uv}\}$, which is the product of $\frac{1-\alpha}{1+\Delta}$ and the optimal value of $LP^H(\boldsymbol{\lambda}, T)$. □

Lemma 14. *For $t \in [\alpha \cdot T, \beta \cdot T)$, the probability of the event E that an offline bin u contains no item before time t is weakly larger than $e^{-\gamma(1+\Delta)\frac{t-\alpha T}{(1-\alpha)T} D}$.*

Proof The packing event of an item of type $v \in V$ into bin u before time t follows a Poisson distribution with a parameter $\lambda_v \frac{\gamma \hat{x}_{uv}}{\hat{\lambda}_v T'}(t-\alpha T)$, where the first term λ_v corresponds to the arrival rate, the second term $\frac{\gamma \hat{x}_{uv}}{\hat{\lambda}_v T'}$ corresponds to the matching probability and the third term $(t-\alpha T)$ corresponds to the time horizon. This is from the property of the compound Poisson process where each random variable is a Bernoulli distribution. From the independency of different online types in V, the event that u gets packed by an online item before time t follows a Poisson distribution with a parameter $\sum_{v\in V} \lambda_v \frac{\gamma \hat{x}_{uv}}{\hat{\lambda}_v T'}(t-\alpha T)$, then we have:

$$\Pr[E] = e^{-\sum_{v\in V} \lambda_v \frac{\gamma \hat{x}_{uv}}{\hat{\lambda}_v T'}(t-\alpha T)} \geq e^{-\gamma(1+\Delta)\frac{t-\alpha T}{(1-\alpha)T}\sum_{v\in V}\hat{x}_{uv}} \geq e^{-\gamma(1+\Delta)\frac{t-\alpha T}{(1-\alpha)T}D}.$$

The first equation is because of the property of Poisson distribution, first inequality is from $\lambda_v \leq (1+\Delta)\cdot\hat{\lambda}_v$ and the second inequality is from Constraint (2b). □

From the above two lemmas, we can come back to prove our lemmas.

Lemma 6. *The probability that an offline bin u contains no item at the end of the heavy LP phase is at least $e^{-\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}D}$. We denote it by q_α^β.*

Proof From Lemma 14, by setting t as βT, we can conclude our statement. □

Lemma 7. *The expected reward during the heavy LP phase is weakly larger than $f_\alpha^\beta \cdot OPT^H$ where $f_\alpha^\beta = \frac{1}{D}(1-3\Delta)(1-\alpha)(1-q_\alpha^\beta)$.*

Proof For each heavy edge pair (u,v) between an offline bin $u\in U$ and an online type $v\in V$, the expected total reward $R_{u,v}^{\text{Heavy LP}}$ contributed by the packing of items of type v into bin u is weakly larger than

$$R_{u,v}^{\text{Heavy LP}} \geq w_{uv}\int_{\alpha T}^{\beta T} e^{-\gamma(1+\Delta)\frac{t-\alpha T}{(1-\alpha)T}D}\cdot\gamma\frac{\hat{x}_{uv}}{\hat{\lambda}_v T'}\cdot\lambda_v dt$$

where inside the integration, the term $e^{-\gamma(1+\Delta)\frac{t-\alpha T}{(1-\alpha)T}D}$ corresponds to the lower bound of the probability of the unmatched event before t (see Lemma 14), the term $\gamma\frac{\hat{x}_{uv}}{\hat{\lambda}_v T'}$ corresponds to the matching probability and $\lambda_v dt$ is the probability that there is a v in the time dt. After integration and simple inequalities, $R_{u,v}^{\text{Heavy LP}}$ can be lower bounded by

$$\begin{aligned}
R_{u,v}^{\text{Heavy LP}} &\geq w_{uv}\hat{x}_{uv}\int_0^{(\beta-\alpha)T}(1-\Delta)\frac{\gamma}{(1-\alpha)T}e^{-\gamma(1+\Delta)\frac{t}{(1-\alpha)T}D}dt\\
&= w_{uv}\hat{x}_{uv}\cdot(1-\Delta)\frac{\gamma}{(1-\alpha)T}\cdot\frac{(1-\alpha)T}{\gamma(1+\Delta)D}\cdot e^{-\gamma(1+\Delta)\frac{t}{(1-\alpha)T}D}\Big|_{(\beta-\alpha)T}^{0}\\
&\geq w_{uv}\hat{x}_{uv}\cdot\frac{1}{D}(1-2\Delta)(1-e^{-\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}D})
\end{aligned}$$

The first inequality is from $(1-\Delta)\cdot\hat{\lambda}_v\leq\lambda_v$, and the last is from $\frac{1-\Delta}{1+\Delta}>1-2\Delta$.

Considering all heavy edge pairs between $u\in U$ and $v\in V$ and applying Lemma 13, the expected reward during the heavy LP phase is

$$\begin{aligned}
\sum_{(u,v)\in\mathcal{E}^H} R_{u,v}^{\text{Heavy LP}} &\geq \frac{1}{D}\frac{1-2\Delta}{1+\Delta}(1-\alpha)(1-e^{-\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}D})\,\mathrm{OPT}^H\\
&\geq \frac{1}{D}(1-3\Delta)(1-\alpha)(1-e^{-\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}D})\,\mathrm{OPT}^H.
\end{aligned}$$

□

B.4. Proof of Lemmas 8 and 9

To prove these lemmas, we also need some auxiliary lemmas. We use w_ℓ to represent the weight of the matching edge ℓ of the arriving vertex at this time t.

LEMMA 15. $\mathbb{E}[w_\ell] \geq \frac{OPT^H}{D\cdot T\sum_{v\in V}\lambda_v}$.

Proof We denote the size of the set V' (Step 16) by k and the total weight of edges in M' by $w(M')$. Since each vertex in V' arrives in the system in the same way, by symmetry, if we fix the size $|V'|$ of the set V' as k', we have $\mathbb{E}[w_\ell|\ |V'|=k'] = \frac{\mathbb{E}[w(M')|\ |V'|=k']}{k'}$.
We then assume the time horizon corresponding to V' is T'. If $T' < T$ corresponding to the case $t < (1-h)T$ in Step 16, we denote the set of the arriving vertices in the following time horizon of $T - T'$ by V''. For each realization of V'', if M'' is the optimal matching of $G'' = (U, V' \cup V'', E)$, $\frac{\mathbb{E}[w(M')|\ |V'|=k']}{k'} \geq \frac{\mathbb{E}[w(M'')|\ |V'|=k', V'']}{k'+|V''|}$, from the maximum weight matching property. We then assume the size of V'' is k'' and take expectations over all V'' with the same size, we have $\frac{\mathbb{E}[w(M')|\ |V'|=k']}{k'} \geq \frac{\mathbb{E}[w(M'')|\ |V'|=k', |V''|=k'']}{k'+k''}$. If we take the expectations over all possible k' and k'', we denote the set of arriving vertices in the time interval $[0,T)$ as V, and we have $\mathbb{E}[w_\ell] \geq \mathbb{E}[\frac{\mathbb{E}[\mathrm{OPT}'|\ |V|=k]}{k}]$, where the outside expectation in the right-hand side if taken over k. Here, OPT' represents the value of the optimal matching. From the fact that there are at most D items in one bin when considering only heavy edges, we have $\mathrm{OPT}' \geq \frac{1}{D}\mathrm{OPT}^H$.
For the second case where $T' = T$ corresponding to $t \geq (1-h)T$ in Step 16, we can directly get $\mathbb{E}[w_\ell|\ |V'| = k'] = \frac{\mathbb{E}[w(M')|\ |V'|=k']}{k'} = \frac{\mathbb{E}[\mathrm{OPT}'|\ |V|=k]}{k}$, because both V and V' is from a time horizon of T and sampled in the same way. We can also get $\mathbb{E}[w_\ell] \geq \mathbb{E}[\frac{\mathbb{E}[\mathrm{OPT}'|\ |V|=k]}{k}]$ by expectation over k.
If suffices to show $\mathbb{E}[\frac{\mathbb{E}[\mathrm{OPT}'|\ |V|=k]}{k}] \geq \frac{\mathrm{OPT}'}{T\sum_{v\in V}\lambda_v}$, which can further show it is at least $\frac{\mathrm{OPT}^H}{D\cdot T\sum_{v\in V}\lambda_v}$. If we treat k as the independent variable x, and we use $f(x)$ to represent the corresponding value $\mathbb{E}[\mathrm{OPT}|\ |V| = x]$, we need to show $\mathbb{E}[\frac{f(x)}{x}] \geq \frac{\mathbb{E}[f(x)]}{\mathbb{E}[x]}$. If we define p_x as the probability of the value x, it's equivalent to show $\sum_x p_x \frac{f(x)}{x} \geq \frac{\sum_x p_x f(x)}{\sum_x p_x x}$. By reformulating the terms, it is equivalent to show $(\sum_x p_x \frac{f(x)}{x})(\sum_x p_x x) \geq \sum_x p_x f(x)$, i.e., $\sum_x p_x^2 f(x) + \sum_{(x,y):x\neq y} p_x p_y \frac{f(x)}{x} y \geq \sum_x p_x f(x)$. By moving the first term in LHS to the right and $1 - p_x = \sum_{y\neq x} p_y$, we get $\sum_{(x,y):x\neq y} p_x p_y \frac{f(x)}{x} y \geq \sum_x p_x (\sum_{y\neq x} p_y) f(x)$.
If we place the two terms considering the same pairs of x and y together, we get:

$$\sum_{\{x,y\}:x\neq y} p_x p_y \left(\frac{f(x)}{x} y + \frac{f(y)}{y} x\right) \geq \sum_{\{x,y\}:x\neq y} p_x p_y (f(x) + f(y)) = \sum_{\{x,y\}:x\neq y} p_x p_y \left(\frac{f(x)}{x} x + \frac{f(y)}{y} y\right)$$

It suffices to show $\frac{f(x)}{x} y + \frac{f(y)}{y} x \geq \frac{f(x)}{x} x + \frac{f(y)}{y} y$ for each pair of x and y, and we can backward all previous equivalent transformations to finish our proof.
Since all arrivals follow independent Poisson processes, when the total number of arrivals is fixed, each arrival's type can be sampled from an i.i.d. distribution according to the parameters of the Poisson process. According to this property and the maximum weight matching property, for the function $f(x) = \mathbb{E}[\mathrm{OPT}|\ |V| = x]$ over all positive integer x, $\frac{f(x)}{x}$ is decreasing with the increase of x. By the rearrangement inequality (Hardy et al. 1952), we can directly get $\frac{f(x)}{x} y + \frac{f(y)}{y} x \geq \frac{f(x)}{x} x + \frac{f(y)}{y} y$. □

LEMMA 16. *When $\beta \cdot T \leq t < \min\{\eta T, (1-h)T\}$, the probability of the event E that bin u is not packed any item before time t is at least $q_\alpha^\beta \cdot \frac{(h+\beta)T}{hT+t}$.*

Proof We denote the size of V' excluding the type v for the present item i as k'. If the total number of arrivals in the time interval $[-hT, \beta T)$ is w, from Lemma 6, we have:

$$\Pr[E|\ w, k'] = q_\alpha^\beta \prod_{each\ arrival\ between\ w\ and\ k'} \Pr[u\ keeps\ unmatched\ upon\ this\ arrival]$$
$$= q_\alpha^\beta \prod_{j=w+1}^{k'} (1 - \frac{1}{j}) \geq q_\alpha^\beta \frac{w}{k'}$$

The second equation is from the symmetry of all arrivals. The last equation is because $\prod_{j=w+1}^{k'} (1 - \frac{1}{j}) = \prod_{j=w+1}^{k'} (\frac{j-1}{j}) = \frac{w}{w+1} \cdot \frac{w+1}{w+2} \cdots \frac{k'-1}{k'} = \frac{w}{k'}$.
We take expectations over all w, we have $\Pr[E|k'] \geq q_\alpha^\beta \frac{\mathbb{E}[w|k']}{k'} = q_\alpha^\beta \frac{(h+\beta)T}{hT+t}$, where the last equation is from the fact that the value of $\frac{\mathbb{E}[w|k']}{k'}$ is exactly equal to the ratio between the length of the time horizon because the probability of an arrival at any time t' in the time interval $[-hT, t)$ is the same. □

LEMMA 17. *If $\beta \leq 1 - h$, when $(1-h)T \leq t < \eta T$, the probability of the event E that bin u is not packed any item before time t is at least $q_\alpha^\beta (h+\beta) e^{-\frac{t-(1-h)T}{T}}$.*

Proof We denote the total number of arrivals in the time interval $[-hT, \beta T)$ and the size of V' excluding the type v for the present item i as w and k' respectively. We further denote the total number of arrivals in the time interval $[(1-h)T, t)$ as num.
Since the event that u is unmatched before time t can be decomposed into three events: u is unmatched before time βT, u is unmatched during the time $[\beta T, (1-h)T)$ and u keeps unmatched during the time $[(1-h)T, t)$. Applying the similar argument in the proof of Lemma 16, the probability of the second event is at least $\frac{w}{k'}$, while the probability of the third event is at least $(1 - \frac{1}{k'+1})^{num}$. Thus, taking expectations over w, we have $\Pr[E|k', num] \geq q_\alpha^\beta (h+\beta)(1 - \frac{1}{k'+1})^{num}$.
From the assumption that T is large, we know that k' is large then $(1 - \frac{1}{k'+1})^{k'+1} \approx e^{-1}$. Then we can get the approximation of $q_\alpha^\beta (h+\beta)(1 - \frac{1}{k'+1})^{num}$:

$$q_\alpha^\beta (h+\beta)(1 - \frac{1}{k'+1})^{num} = q_\alpha^\beta (h+\beta)(1 - \frac{1}{k'+1})^{(k'+1)\cdot \frac{num}{k'+1}} \approx q_\alpha^\beta (h+\beta) e^{-\frac{num}{k'+1}}.$$

If we treat $\frac{num}{k'+1}$ as a whole, because of the convexity of the function e^{-x}, $\Pr[E] \geq q_\alpha^\beta (h+\beta)\mathbb{E}[e^{-\frac{num}{k'+1}}] \geq q_\alpha^\beta (h+\beta) e^{-\mathbb{E}[\frac{num}{k'+1}]}$. It suffices to show $\mathbb{E}[\frac{num}{k'+1}] \leq \frac{t-(1-h)T}{T}$.
From the independency between num and k', $\mathbb{E}[\frac{num}{k'+1}] = \mathbb{E}[num]\mathbb{E}[\frac{1}{k'+1}]$. For $\mathbb{E}[num]$, it is equal to $(t-(1-h)T)\sum_{v\in V} \lambda_v$. For $\mathbb{E}[\frac{1}{k'+1}]$, we can treat the distribution of k' as a Poisson distribution with some fixed parameter λ' because of the additive property of independent Poisson distribution. We have:

$$\mathbb{E}\left[\frac{1}{k'+1}\right] = \sum_{x=0}^{\infty} \frac{(\lambda')^x}{x!} e^{-\lambda'} \frac{1}{x+1} = \frac{1}{\lambda'} \sum_{x=0}^{\infty} \frac{(\lambda')^{x+1}}{(x+1)!} e^{-\lambda'} = \frac{1}{\lambda'}(1 - e^{-\lambda'}) \leq \frac{1}{\lambda'} = \frac{1}{\mathbb{E}[k']}.$$

Thus, we get $\mathbb{E}[\frac{num}{k'+1}] = \mathbb{E}[num]\mathbb{E}[\frac{1}{k'+1}] \leq \frac{(t-(1-h)T)\sum_{v\in V}\lambda_v}{T\sum_{v\in V}\lambda_v}$, which is equal to the wanted term. □

LEMMA 18. *If $\beta > 1 - h$, when $\beta T \leq t < \eta T$, the probability of the event E that bin u is not packed any item before time t is at least $q_\alpha^\beta e^{-\frac{t-\beta T}{T}}$.*

Proof We denote the size of V' excluding the type v for the present item i and the total number of arrivals in the time interval $[\beta T, t)$ as k' and num, respectively.
Since the event that u is unmatched before time t can be decomposed into two events: u is unmatched before time βT and u keeps unmatched during the time $[\beta T, t)$. Applying the similar argument in the proof of Lemma 17, the probability of the second event is at least $(1-\frac{1}{k'+1})^{num}$.Hence we have $\Pr[E|k', num] \geq q_\alpha^\beta(1-\frac{1}{k'+1})^{num}$. From the assumption that T is large, $q_\alpha^\beta(1-\frac{1}{k'+1})^{num} = q_\alpha^\beta(1-\frac{1}{k'+1})^{(k'+1)\cdot\frac{num}{k'+1}} \approx q_\alpha^\beta e^{-\frac{num}{k'+1}}$. If we treat $\frac{num}{k'+1}$ as a whole, because of the convexity of the function e^{-x}, $\Pr[E] \geq q_\alpha^\beta \mathbb{E}[e^{-\frac{num}{k'+1}}] \geq q_\alpha^\beta e^{-\mathbb{E}[\frac{num}{k'+1}]}$. It suffices to show $\mathbb{E}[\frac{num}{k'+1}] \leq \frac{t-\beta T}{T}$. The term $\mathbb{E}[\frac{num}{k'+1}]$ can be upper bounded by $\mathbb{E}[num]\frac{1}{\mathbb{E}[k']}$ following the same procedure in the proof of Lemma 17. □
We now come back to the proof of Lemmas 8 and 9.

Lemma 8. *The expected reward during the heavy maximum matching phase is at least $q_\alpha^\beta f_\beta^\eta \cdot OPT^H$, where f_β^η is defined as:*

$$f_\beta^\eta = \begin{cases} \frac{1}{D}(h+\beta)\ln\frac{h+\eta}{h+\beta}, & \eta \leq 1-h \\ \frac{1}{D}(h+\beta)(\ln\frac{1}{h+\beta}+1-e^{1-h-\eta}), & \eta > 1-h \geq \beta \\ \frac{1}{D}(1-e^{-(\eta-\beta)}), & \beta > 1-h. \end{cases}$$

Proof By applying Lemma 15, the expected reward $R^{\text{Heavy max}}$ during the maximum matching phase is at least

$$R^{\text{Heavy max}} \geq \int_{\beta T}^{\eta T} \sum_{v\in V} \lambda_v \cdot \frac{\text{OPT}^H}{D\cdot T\sum_{v\in V}\lambda_v} \cdot \Pr[E]dt = \frac{\text{OPT}^H}{D}\int_{\beta T}^{\eta T}\frac{1}{T}\cdot\Pr[E]dt,$$

where E is the event that the corresponding offline vertex u of the online item i arriving at time t is unmatched before time t. We can utilize Lemma 16 for $\eta \leq 1-h$, Lemmas 16 and 17 for $\beta \leq 1-h < \eta$ and Lemma 18 for $\beta > 1-h$ to get the required lower bound of $\Pr[E]$. Specifically,

$$\Pr[E] \geq \begin{cases} q_\alpha^\beta \cdot \frac{(h+\beta)T}{hT+t}, & \eta \leq 1-h \\ q_\alpha^\beta \cdot \frac{(h+\beta)T}{hT+t}, & \eta > 1-h \geq \beta, \beta T \leq t < (1-h)T \\ q_\alpha^\beta(h+\beta)e^{-\frac{t-(1-h)T}{T}}, & \eta > 1-h \geq \beta, (1-h)T \leq t < \eta T \\ q_\alpha^\beta e^{-\frac{t-\beta T}{T}}, & \beta > 1-h. \end{cases}$$

Then, we calculate the corresponding integration value and get the result.
When $\eta \leq 1-h$, we have:

$$\begin{aligned} R^{\text{Heavy max}} &\geq \frac{1}{D}\text{OPT}^H\int_{\beta T}^{\eta T}\frac{1}{T}q_\alpha^\beta\frac{(h+\beta)T}{hT+t}dt \\ &= q_\alpha^\beta\frac{1}{D}\text{OPT}^H\left((h+\beta)\ln(hT+t)\right)\Big|_{\beta T}^{\eta T} \\ &= q_\alpha^\beta\frac{1}{D}(h+\beta)\ln\frac{h+\eta}{h+\beta}\text{OPT}^H. \end{aligned}$$

When $\eta > 1-h \geq \beta$, we have:

$$\begin{aligned} R^{\text{Heavy max}} &\geq \frac{1}{D}\text{OPT}^H\left(\int_{\beta T}^{(1-h)T}\frac{1}{T}q_\alpha^\beta\frac{(h+\beta)T}{hT+t}dt + \int_{(1-h)T}^{\eta T}\frac{1}{T}q_\alpha^\beta(h+\beta)e^{-\frac{t-(1-h)T}{T}}dt\right) \\ &= q_\alpha^\beta\frac{1}{D}(h+\beta)\text{OPT}^H\left(\int_{\beta T}^{(1-h)T}\frac{1}{T}\frac{T}{hT+t}dt + \int_{(1-h)T}^{\eta T}\frac{1}{T}e^{-\frac{t-(1-h)T}{T}}dt\right) \\ &= q_\alpha^\beta\frac{1}{D}(h+\beta)\text{OPT}^H\left(\left(\ln(hT+t)\right)\Big|_{\beta T}^{(1-h)T} + \left(-e^{-\frac{t-(1-h)T}{T}}\right)\Big|_{(1-h)T}^{\eta T}\right) \\ &= q_\alpha^\beta\frac{1}{D}(h+\beta)\left(\ln\frac{1}{h+\beta}+1-e^{1-h-\eta}\right)\text{OPT}^H. \end{aligned}$$

When $\beta > 1-h$, we have:

$$\begin{aligned} R^{\text{Heavy max}} &\geq \frac{1}{D}\,\text{OPT}^H \int_{\beta T}^{\eta T} \frac{1}{T} q_\alpha^\beta e^{-\frac{t-\beta T}{T}}\, dt \\ &= q_\alpha^\beta \frac{1}{D}\,\text{OPT}^H \int_{\beta T}^{\eta T} \frac{1}{T} e^{-\frac{t-\beta T}{T}}\, dt \\ &= q_\alpha^\beta \frac{1}{D}\,\text{OPT}^H \left(-e^{-\frac{t-\beta T}{T}}\right)\Big|_{\beta T}^{\eta T} \\ &= q_\alpha^\beta \frac{1}{D}(1-e^{-(\eta-\beta)})\,\text{OPT}^H . \end{aligned}$$

□

Lemma 9. *Given the fact that bin u contains no items before the heavy maximum matching phase, the probability of the event E that bin u is not packed any item during the heavy maximum matching phase is at least*

$$q_\beta^\eta = \begin{cases} \frac{h+\beta}{h+\eta}, & \eta \leq 1-h \\ (h+\beta)e^{1-h-\eta}, & \eta > 1-h \geq \beta \\ e^{-(\eta-\beta)}, & \beta > 1-h. \end{cases}$$

Proof By applying Lemmas 16, 17 and 18, conditioning on the event that bin u contains no items before the heavy maximum matching phase, we can set t as ηT and get our statement. □

B.5. Proof of Lemmas 10 and 11

Before the proof, we first present two additional lemmas.

Lemma 19. *The expectation of the optimal value of $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C})$ is lower bounded by $q_\alpha^\beta q_\beta^\eta \frac{1-\eta}{1+\Delta'}\, OPT^L$.*

Proof From Lemma 4, the optimal value of $LP_0^L(\boldsymbol{\lambda}, T, C)$ is an upper bound of the offline optimal OPT in the original problem. It suffices to show the expectation of the optimal value of $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C})$ is lower bounded by the product of $q_\alpha^\beta q_\beta^\eta \frac{1-\eta}{1+\Delta'}$ and the optimal value of $LP_0^L(\boldsymbol{\lambda}, T, C)$. We assume the optimal solution of $LP_0^L(\boldsymbol{\lambda}, T, C)$ is $\{y_{uv}^*\}$. We next build the corresponding solution of $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C})$ for each realization $\bar{C}$ to show the statement.

For a realization $\bar{C}$, we set the value of $\{y_{uv}\}$ for each bin u. If the bin u contains no item before the light phase, we set $\frac{1-\eta}{1+\Delta'} y_{uv}^*$ for each $v \in V$ as the value of the corresponding y_{uv} of $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C})$. Otherwise, if the bin u contains some items before the light phase, we set 0 for each v as the value of the corresponding y_{uv}.

We can first check such a solution is feasible for $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C})$. For Constraints (3a), $\sum_{u\in U,(u,v)\in\mathcal{E}^L} y_{uv} \leq \sum_{u\in U,(u,v)\in\mathcal{E}^L} \frac{1-\eta}{1+\Delta'} y_{uv}^* \leq \frac{1-\eta}{1+\Delta'}\lambda_v T \leq \hat{\lambda}_v T'$. Here, the first inequality is from the fact that some values will be set as 0, while the second inequality is from Constraints (3a) for $LP_0^L(\boldsymbol{\lambda}, T, C)$. The last inequality is from the definition of T' and $\hat{\boldsymbol{\lambda}}$.

For Constraints (3b), since only the value for the bin u that contains no item before the light phase can remain nonzero, where $\bar{C}_u^d = C_u^d$, and the coefficient $\frac{1-\eta}{1+\Delta'} \leq 1$, by Constraints (3b) for $LP_0^L(\boldsymbol{\lambda}, T, C)$, these constraints hold under our given solution.

Now we can compare the optimal values. Assuming a set U' which contains only the bins without heavy items, $\Pr[U']$ is the corresponding probability, then the expectation of the optimal value of $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C})$ is equal to

$$\sum_{U'} \Pr[U'] \sum_{u\in U', v\in V, (u,v)\in\mathcal{E}^L} w_{uv} \frac{1-\eta}{1+\Delta'} y^*_{uv} = \frac{1-\eta}{1+\Delta'} \sum_{u\in U, v\in V, (u,v)\in\mathcal{E}^L} w_{uv} y^*_{uv} \sum_{U': u\in U'} \Pr[U']$$
$$\geq q_\alpha^\beta q_\beta^\eta \frac{1-\eta}{1+\Delta'} \, \mathrm{OPT}^L .$$

The inequality is from Lemma 4 and the definition of $q_\alpha^\beta q_\beta^\eta$ in Lemmas 6 and 9. □

LEMMA 20. *Conditioning on an offline bin u is not packed any item before time $\eta\cdot T$, for $t\in[\eta\cdot T, \theta\cdot T)$, the probability of the event E that u is available after $\eta\cdot T$ before time t is weakly larger than $1-2D\gamma'(1+\Delta')\frac{t-\eta T}{(1-\eta)T}$.*

Proof Let Z^d_{ut} denote the consumed capacity of u's d-th dimension by time t.

$$\begin{aligned}
\Pr[E] &= \Pr[\wedge_{d\in[D]} (C^d_u - Z^d_{ut}) \geq \frac{1}{2} C^d_u] \\
&= 1 - \Pr[\neg \wedge_{d\in[D]} (C^d_u - Z^d_{ut}) \geq \frac{1}{2} C^d_u] \\
&= 1 - \Pr[\vee_{d\in[D]} (C^d_u - Z^d_{ut}) < \frac{1}{2} C^d_u] \\
&\geq 1 - \sum_{d\in[D]} \Pr[(C^d_u - Z^d_{ut}) < \frac{1}{2} C^d_u] \\
&\geq 1 - \sum_{d\in[D]} \Pr[Z^d_{ut} > \frac{1}{2} C^d_u]
\end{aligned}$$

Here $\wedge, \neg, \vee$ are logical "and", "not" and "or". According to Markov's inequality, we have

$$\Pr[Z^d_{ut} > \frac{1}{2} C^d_u] \leq \frac{\mathbb{E}[Z^d_{ut}]}{\frac{1}{2} C^d_u}.$$

By the matching probability $\gamma' \cdot \frac{\hat{y}_{uv}}{\hat{\lambda}_v T'}$ given in Step 28 of Algorithm 1, with Constraints (3b), we have

$$\mathbb{E}[Z^d_{ut}] = \sum_{v\in V} \lambda_v (t-\eta T) r^d_{uv} \gamma' \cdot \frac{\hat{y}_{uv}}{\hat{\lambda}_v T'} \leq (t-\eta T) \frac{\gamma'(1+\Delta')}{(1-\eta)T} \sum_v \hat{y}_{uv} r^d_{uv} \leq (t-\eta T) \frac{\gamma'(1+\Delta')}{(1-\eta)T} C^d_u$$

Then combining all inequalities before, we can further bound the $\Pr[E]$:

$$\begin{aligned}
\Pr[E] &\geq 1 - \sum_{d\in[D]} \Pr[Z^d_{ut} > \frac{1}{2} C^d_u] \\
&\geq 1 - \sum_{d\in[D]} \frac{\mathbb{E}[Z^d_{ut}]}{\frac{1}{2} C^d_u} \\
&\geq 1 - \sum_{d\in[D]} 2\gamma'(1+\Delta') \frac{t-\eta T}{(1-\eta)T} \\
&\geq 1 - 2D\gamma'(1+\Delta') \frac{t-\eta T}{(1-\eta)T}.
\end{aligned}$$

□

We now prove our main lemmas in this section.

LEMMA 10. *The expected reward during the light LP phase is at least $q_\alpha^\beta q_\beta^\eta f_\eta^\theta \cdot OPT^L$, where f_η^θ is defined as below:*

$$f_\eta^\theta = (1-2\Delta')\gamma'(\theta-\eta) \left(1 - D\gamma'(1+\Delta') \frac{\theta-\eta}{1-\eta}\right).$$

Proof From the solution used in the proof of Lemma 19, under a given U' which contains all available bins at the beginning of the light LP phase, we want to show our algorithm can reach a comparable guarantee according to that solution. For a given U', the expected reward during the light LP phase $R^{\text{Light LP}}$ is at least

$$R^{\text{Light LP}} \geq \int_{\eta T}^{\theta T} \sum_{v} \left(\lambda_v \sum_{u \in U'} w_{uv} \cdot \gamma' \frac{\hat{y}_{uv}}{\hat{\lambda}_v T'} \cdot (1 - 2D\gamma'(1+\Delta') \frac{t - \eta T}{(1-\eta)T}) \right) dt.$$

Here, the term $\gamma' \frac{\hat{y}_{uv}}{\hat{\lambda}_v T'}$ is the matching probability, and the term $(1 - 2D\gamma'(1+\Delta')\frac{t-\eta T}{(1-\eta)T})$ is the available probability of the bin u from Lemma 20. With some calculus, we can calculate its value, which is:

$$\begin{aligned} R^{\text{Light LP}} &\geq \sum_{v \in V, u \in U', (v,u) \in \mathcal{E}^L} w_{uv} \hat{y}_{uv} \frac{\lambda_v}{\hat{\lambda}_v} \gamma' \int_{\eta T}^{\theta T} \frac{1}{T'} (1 - 2D\gamma'(1+\Delta') \frac{t - \eta T}{(1-\eta)T}) dt \\ &\geq \sum_{v \in V, u \in U', (v,u) \in \mathcal{E}^L} w_{uv} \hat{y}_{uv} (1 - \Delta')\gamma' \left(\frac{1}{(1-\eta)T} (t - D\gamma'(1+\Delta') \frac{(t-\eta T)^2}{(1-\eta)T}) \right) \Bigg|_{\eta T}^{\theta T} \\ &= \sum_{v \in V, u \in U', (v,u) \in \mathcal{E}^L} w_{uv} \hat{y}_{uv} (1 - \Delta')\gamma' \frac{\theta - \eta}{1 - \eta} (1 - D\gamma'(1+\Delta') \frac{\theta - \eta}{1 - \eta}). \end{aligned}$$

Since the optimal value of $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C})$ is exactly $\sum_{v \in V, u \in U', (v,u) \in \mathcal{E}^L} w_{uv} \hat{y}_{uv}$ and $LP_0^L(\hat{\boldsymbol{\lambda}}, T', \bar{C}) \geq q_\alpha^\beta q_\beta^\eta \frac{1-\eta}{1+\Delta'} \mathrm{OPT}^L$ (Lemma 19), we can further lower bound $R^{\text{Light LP}}$:

$$\begin{aligned} R^{\text{Light LP}} &\geq \sum_{v \in V, u \in U', (v,u) \in \mathcal{E}^L} w_{uv} \hat{y}_{uv} (1 - \Delta')\gamma' \frac{\theta - \eta}{1 - \eta} \left(1 - D\gamma'(1+\Delta') \frac{\theta - \eta}{1 - \eta}\right) \\ &\geq (1 - \Delta')\gamma' \frac{\theta - \eta}{1 - \eta} \left(1 - D\gamma'(1+\Delta') \frac{\theta - \eta}{1 - \eta}\right) q_\alpha^\beta q_\beta^\eta \frac{1-\eta}{1+\Delta'} \mathrm{OPT}^L \\ &\geq q_\alpha^\beta q_\beta^\eta f_\eta^\theta \cdot \mathrm{OPT}^L \end{aligned}$$

where $f_\eta^\theta = (1 - 2\Delta')\gamma'(\theta - \eta)\left(1 - D\gamma'(1+\Delta')\frac{\theta-\eta}{1-\eta}\right)$. Last inequality is because $\frac{1-\Delta'}{1+\Delta'} \geq 1 - 2\Delta'$. □

Lemma 11. *Conditioning on the event that bin u has not been packed any item before the light LP phase, the expected consumption of this bin in d-th dimension for each $d \in [D]$ is at most $\gamma'(1+\Delta')\frac{\theta-\eta}{1-\eta} C_u^d$. That is, $q_\eta^\theta = \gamma'(1+\Delta')\frac{\theta-\eta}{1-\eta}$.*

Proof From the upper bound of the term $\mathbb{E}[Z_{ut}^d]$ given in the proof of Lemma 20, by setting $t = \theta T$, we can prove our statement. □

B.6. Proof of Lemma 12

This lemma holds from the following lemmas.

Lemma 21. $\mathbb{E}[w_\ell] \geq \frac{OPT^L}{T \sum_{v \in V} \lambda_v}$.

Proof Before we start our proof, we first need to show the following claim holds.

Claim. If we denote the total number of arrivals during a time horizon and the expected value of LP_1^L given the total number of arrivals is x and $f(x)$, respectively, then $\frac{f(x)}{x}$ is a decreasing function.

Such property holds because of the following reasons. Since the number of arrivals are fixed, the Poisson arrival model can be seen as following independent and identical distributions, where the probability of each type v is proportional to the arrival rate λ_v. Under this model, since all arrivals are identical in expectations, it suffices to show removing the last arrival can still satisfy the constraints of LP_1^L. For an optimal solution $\{y_{uv}\}$ of $LP_1^L(V')$, we can show $\{y_{uv}: v \neq v'\}$ is a feasible solution of $LP_1^L(V' - \{v'\})$, where v' is the last arrival. This obviously holds for Constraints (4a) and (4c). For Constraints (4b), only the left-hand side can be decreased, so they still hold. We now finish the proof of the claim.

For the lemma, we adopt the ideas behind the proof of Lemma 15 and prove our statements. We denote the size of the set V' (Step 33) by k and the optimal value of LP (4) by OPT$'$. Since each vertex in V' arrives in the system in the same way, by symmetry, if we fix the size $|V'|$ of the set V' as k', we have $\mathbb{E}[w_\ell|\ |V'|=k'] = \frac{\mathbb{E}[\mathrm{OPT}'|\ |V'|=k']}{k'}$.

We then assume the time horizon corresponding to V' is T'. If $T' < T$ corresponding to the case $t < (1-h)T$ in Step 33, we denote the set of the arriving vertices in the following time horizon of $T - T'$ by V''. For each realization of V'', if OPT$''$ is the optimal value of $LP_1^L(V' \cup V'')$, $\frac{\mathbb{E}[\mathrm{OPT}'|\ |V'|=k']}{k'} \geq \frac{\mathbb{E}[\mathrm{OPT}''|\ |V'|=k', V'']}{k'+|V''|}$, from the claim above. We then assume the size of V'' is k'' and take expectations over all V'' with the same size, we have $\frac{\mathbb{E}[\mathrm{OPT}'|\ |V'|=k']}{k'} \geq \frac{\mathbb{E}[\mathrm{OPT}''|\ |V'|=k', |V''|=k'']}{k'+k''}$. If we take the expectations over all possible k' and k'', we denote the set of arriving vertices in the time interval $[0, T)$ as V, and we have $\mathbb{E}[w_\ell] \geq \mathbb{E}[\frac{\mathbb{E}[\mathrm{OPT}^L|\ |V|=k]}{k}]$, where the outside expectation in the right-hand side if taken over k. This is because OPT^L corresponds to the optimal value of LP_1^L in a time horizon of exactly T. For the second case where $T' = T$ corresponding to $t \geq (1-h)T$ in Step 33, we can directly get $\mathbb{E}[w_\ell|\ |V'| = k'] = \frac{\mathbb{E}[\mathrm{OPT}'|\ |V'|=k']}{k'} = \frac{\mathbb{E}[\mathrm{OPT}^L|\ |V|=k]}{k}$, because both V and V' is from a time horizon of T and sampled in the same way. We can also get $\mathbb{E}[w_\ell] \geq \mathbb{E}[\frac{\mathbb{E}[\mathrm{OPT}^L|\ |V|=k]}{k}]$ by expectation over k.

If suffices to show $\mathbb{E}[\frac{\mathbb{E}[\mathrm{OPT}^L|\ |V|=k]}{k}] \geq \frac{\mathrm{OPT}^L}{T\sum_{v\in V}\lambda_v}$. If we treat k as the independent variable x and use the above function $f(x)$, following the same procedure in Lemma 15, we finish our proof. □

LEMMA 22. *When $\theta T \leq t < T$, the probability of the event E that the corresponding edge ℓ chosen by Step 34 in Algorithm 1 can be matched successfully conditioning on the corresponding bin u is not packed by any item before ηT is at least*

$$\begin{cases} 1 - 2D(\ln \frac{hT+t}{hT+\theta T} + q_\eta^\theta), & \theta T \leq t < (1-h)T, \theta \leq 1-h \\ 1 - 2D(\ln \frac{1}{h+\theta} + \frac{t-(1-h)T}{T} + q_\eta^\theta), & (1-h)T \leq t < T, \theta \leq 1-h \\ 1 - 2D(\frac{t-\theta T}{T} + q_\eta^\theta), & \theta T \leq t < T, \theta > 1-h. \end{cases}$$

Proof We can adopt some ideas used in the proof of Lemmas 16, 17 and 20 to prove this statement. For the first case where $\theta \leq 1-h$ and $\theta T \leq t < (1-h)T$, we assume the total number of arrivals in time $[-hT, \theta T)$ is w and the number of arrivals in time $[-hT, t)$ is k. Because all arrivals are symmetric, we can conclude the expectation of the consumption of bin u in the d-th dimension after the light LP phase is no greater than $\sum_{i=w+1}^{k} \frac{C_u^d}{i}$. Combining with the expectation of the consumption during the light LP phase shown in Lemma 11, the expectation of the total consumption is at most $\sum_{i=w+1}^{k} \frac{C_u^d}{i} + q_\eta^\theta C_u^d$. By applying the union bound and Markov's inequality, the probability of the event E' (which can imply E) that the consumption of bin u in each dimension does not exceed a half of the corresponding capacity entry is at least $\mathbb{E}_{k,w}[1 - \sum_{i=w+1}^{k} \frac{2D}{i} - 2Dq_\eta^\theta] \geq 1 - 2D(\ln \frac{hT+t}{hT+\theta T} + q_\eta^\theta)$. The inequality is because the sum can be upper bounded by $ln \frac{k}{w}$, where the logarithmic function is a concave function and the expectation of $\frac{k}{w}$ is the ratio between the corresponding time horizon, by the symmetric argument used in the proof of Lemma 16.

For the second case where $\theta \le 1-h$ and $(1-h)T \le t < T$. We assume the total number of arrivals in time $[-hT, \theta T)$ is w, the number of arrivals in time $[-hT, (1-h)\cdot T)$ is k and the number of arrivals in time $[(1-h)\cdot T, t)$ is num. We can also conclude that the expectation of the consumption of bin u in the d-th dimension after the light LP phase is upper bounded by $\sum_{i=w+1}^{k} \frac{C_u^d}{i} + \frac{num\cdot C_u^d}{k+1}$. Thus, the total consumption is at most $\sum_{i=w+1}^{k} \frac{C_u^d}{i} + \frac{num\cdot C_u^d}{k+1} + q_\eta^\theta C_u^d$. Again with union bound and Markov's inequality, the probability of the event E' is at least $\mathbb{E}_{k,w,num}[1-\sum_{i=w+1}^{k} \frac{2D}{i} - \frac{2D\cdot num}{k+1} - 2Dq_\eta^\theta] \ge 1-2D(\ln\frac{1}{h+\theta} + \frac{t-(1-h)T}{T} + q_\eta^\theta)$. Here, the transformation from the term $\frac{num}{k+1}$ to $\frac{t-(1-h)T}{T}$ follows the same ideas as in the proof of Lemma 17.

For the last case where $\theta T \le t < T$ and $\theta > 1-h$. We assume the number of arrivals in time $[-hT, (1-h)\cdot T)$ is k and the number of arrivals in time $[\theta \cdot T, t)$ is num. With the similar idea above, the total consumption is at most $\frac{num\cdot C_u^d}{k+1} + q_\eta^\theta C_u^d$. Thus, the probability of the event E' is at least $\mathbb{E}_{k,num}[1 - \frac{2D\cdot num}{k+1} - 2Dq_\eta^\theta] \ge 1-2D(\frac{t-\theta T}{T} + q_\eta^\theta)$. □

According to the lemmas above, we can prove Lemma 12.

LEMMA 12. *The expected reward during the light maximum packing phase is at least $q_\alpha^\beta q_\beta^\eta f_\theta^1 \cdot OPT^L$ where f_θ^1 is*

$$f_\theta^1 = \begin{cases} (1+2D)(1-h-\theta) + 2D\ln(h+\theta) + h(1+2D\ln(h+\theta) - Dh) - 2D(1-\theta)q_\eta^\theta, & \theta \le 1-h \\ (1-\theta)(1-D(1-\theta)) - 2D(1-\theta)q_\eta^\theta, & \theta > 1-h. \end{cases}$$

Proof The expected reward $R^{\text{Light max}}$ during this phase can be calculated by

$$R^{\text{Light max}} = \int_{\theta T}^{T} \sum_{v\in V} \lambda_v \cdot \frac{\text{OPT}^L}{T\sum_v \lambda_v} \cdot q_\alpha^\beta q_\beta^\eta \Pr[E]dt = q_\alpha^\beta q_\beta^\eta \,\text{OPT}^L \int_{\theta T}^{T} \frac{1}{T} \cdot \Pr[E]dt.$$

Here, the term $\sum_{v\in V}\lambda_v$ represents the arrival rate for any online type, the term $\frac{\text{OPT}^L}{T\sum_v \lambda_v}$ is the bound of the weight of one matching edge from Lemma 21, and the last term $q_\alpha^\beta q_\beta^\eta \Pr[E]$ represents the probability that the chosen edge can be matched successfully. $\Pr[E]$ is the probability mentioned in Lemma 22, which should be further categorized for the following calculation. According to the corresponding lower bound of $\Pr[E]$, we can calculate $R^{\text{Light max}}$. For the first case when $\theta \le 1-h$, we have:

$$\begin{aligned} &R^{\text{Light max}} \\ &\ge q_\alpha^\beta q_\beta^\eta \,\text{OPT}^L \left(\int_{\theta T}^{(1-h)T} \frac{1}{T}(1-2D(\ln\frac{hT+t}{hT+\theta T} + q_\eta^\theta))dt + \int_{(1-h)T}^{T} \frac{1}{T}(1-2D(\ln\frac{1}{h+\theta} + \frac{t-(1-h)T}{T} + q_\eta^\theta))dt \right) \\ &= q_\alpha^\beta q_\beta^\eta \,\text{OPT}^L \left(\left(\frac{t}{T}(1-2Dq_\eta^\theta) - 2D\frac{hT+t}{T}(\ln\frac{hT+t}{hT+\theta T} - 1)\right)\Bigg|_{\theta T}^{(1-h)T} \right. \\ &\qquad \left. + \left(\frac{t}{T}(1-2D(\ln\frac{1}{h+\theta} + q_\eta^\theta)) - D\frac{(t-(1-h)T)^2}{T^2}\right)\Bigg|_{(1-h)T}^{T} \right) \\ &= q_\alpha^\beta q_\beta^\eta \,\text{OPT}^L \left((1-2Dq_\eta^\theta)(1-\theta) + 2Dh\ln(h+\theta) - 2D(\ln\frac{1}{h+\theta} - 1) - 2D(h+\theta) \right) \\ &= q_\alpha^\beta q_\beta^\eta \left(1+2D)(1-h-\theta) + 2D\ln(h+\theta) + h(1+2D\ln(h+\theta) - Dh) - 2D(1-\theta)q_\eta^\theta\right) \text{OPT}^L. \end{aligned}$$

For the second case when $\theta > 1-h$, we have:

$$R^{\text{Light max}} \ge q_\alpha^\beta q_\beta^\eta \,\text{OPT}^L \left(\int_{\theta T}^{T} \frac{1}{T}\left(1-2D(\frac{t-\theta T}{T} + q_\eta^\theta)\right) dt \right)$$

$$
\begin{aligned}
&= q_\alpha^\beta q_\beta^\eta \, \mathrm{OPT}^L \left(\frac{t}{T}(1 - 2Dq_\eta^\theta) - D\frac{(t-\theta T)^2}{T^2} \right)\Bigg|_{\theta T}^{T} \\
&= q_\alpha^\beta q_\beta^\eta \, \mathrm{OPT}^L((1-\theta)(1-2Dq_\eta^\theta) - D(1-\theta)^2) \\
&= q_\alpha^\beta q_\beta^\eta((1-\theta)(1-D(1-\theta)) - 2D(1-\theta)q_\eta^\theta)\, \mathrm{OPT}^L .
\end{aligned}
$$

□

B.7. Proof of Theorem 1

THEOREM 1. *Denote $N = \min_{v\in V} \lambda_v \cdot T$. For $0 < \delta < 1$, by choosing phase parameters α, β, η and θ, and scaling parameters γ and γ' satisfying $0 \le \alpha \le \beta \le \eta \le \theta \le 1$ and $0 \le \gamma, \gamma' \le 1$, with probability at least $1 - 2m\delta - me^{\frac{-(h+\alpha)N}{8}} - me^{\frac{-(h+\eta)N}{8}}$, Algorithm 1 has a competitive ratio of at least*

$$
\max_{\alpha,\beta,\eta,\theta,\gamma,\gamma'} \min\{F^H, F^L\}
$$

where

$$
F^H = f_\alpha^\beta + \begin{cases} q_\alpha^\beta \cdot \frac{1}{D}(h+\beta)\ln\frac{h+\eta}{h+\beta}, & \eta \le 1-h \\ q_\alpha^\beta \cdot \frac{1}{D}(h+\beta)(\ln\frac{1}{h+\beta} + 1 - e^{1-h-\eta}), & \beta \le 1-h < \eta \\ q_\alpha^\beta \cdot \frac{1}{D}(1 - e^{-(\eta-\beta)}), & 1-h < \beta, \end{cases}
$$

$$
F^L = \begin{cases} q_\alpha^\beta \cdot \frac{h+\beta}{h+\eta} \cdot \left(f_\eta^\theta + f_1\right), & \theta \le 1-h \\ q_\alpha^\beta \cdot \frac{h+\beta}{h+\eta} \cdot \left(f_\eta^\theta + f_2\right), & \eta \le 1-h < \theta \\ q_\alpha^\beta \cdot (h+\beta)e^{1-h-\eta} \cdot \left(f_\eta^\theta + f_2\right), & \beta \le 1-h < \eta \\ q_\alpha^\beta \cdot e^{-(\eta-\beta)} \cdot \left(f_\eta^\theta + f_2\right), & 1-h < \beta. \end{cases}
$$

Here

$$
\begin{cases} f_\alpha^\beta = \frac{1}{D}(1-3\Delta)(1-\alpha)(1-q_\alpha^\beta), \\ q_\alpha^\beta = e^{-\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}D}, \Delta = \sqrt{\frac{8ln\frac{1}{\delta}}{(h+\alpha)N}}, \end{cases}
$$

$$
\begin{cases} f_\eta^\theta = (1-2\Delta')\gamma'(\theta-\eta)(1 - D\gamma'(1+\Delta')\frac{\theta-\eta}{1-\eta}), \\ q_\eta^\theta = \gamma'(1+\Delta')\frac{\theta-\eta}{1-\eta}, \Delta' = \sqrt{\frac{8ln\frac{1}{\delta}}{(h+\eta)N}}, \end{cases}
$$

$$
\begin{cases} f_1 = (1+2D)(1-h-\theta) + 2D\ln(h+\theta) + h(1 + 2D\ln(h+\theta) - Dh) - 2D(1-\theta)q_\eta^\theta, \\ f_2 = (1-\theta)(1-D(1-\theta)) - 2D(1-\theta)q_\eta^\theta. \end{cases}
$$

Proof Since we can directly summarize Lemmas 5, 7, 8, 10 and 12 to get the values of competitive ratio under different choices of the phase parameters, it suffices to show the assumptions $(1-\Delta)\cdot\hat{\lambda}_v \le \lambda_v \le (1+\Delta)\cdot\hat{\lambda}_v$ and $(1-\Delta')\cdot\hat{\lambda}_v \le \lambda_v \le (1+\Delta')\cdot\hat{\lambda}_v$ are satisfied in the corresponding phases with the given probability in the statement of the theorem.

For the heavy LP phase, the assumption that $(1-\Delta)\cdot\hat{\lambda}_v \le \lambda_v \le (1+\Delta)\cdot\hat{\lambda}_v$ holds for all $v \in V$ is needed. From Lemma 1, we get the event that the number of arrivals of type v is at least $\frac{1}{2}(h+\alpha)\lambda_v T$ for a $v \in V$ holds with a probability of at least $1 - e^{-\frac{\lambda_v(h+\alpha)T}{8}} \ge 1 - e^{-\frac{(h+\alpha)N}{8}}$. Conditioning on the event above, from Lemma 2, with probability $1-\delta$, $(1-c)\cdot\hat{\lambda}_v \le \lambda_v \le (1+c)\cdot\hat{\lambda}_v$ holds for a $v \in V$, where $c = \sqrt{\frac{4\ln\frac{1}{\delta}}{n}} \le \sqrt{\frac{4\ln\frac{1}{\delta}}{\frac{1}{2}(h+\alpha)\lambda_v T}} \le \sqrt{\frac{8\ln\frac{1}{\delta}}{(h+\alpha)N}} = \Delta$. By union bound over all $v \in V$, we can finally get the assumption holds for the heavy LP phase with a probability of at least $1 - m\delta - me^{-\frac{(h+\alpha)N}{8}}$.

The case for the light LP phase is similar, where we replace Δ and α by Δ' and η respectively, and then we get the assumption $(1-\Delta')\cdot\hat{\lambda}_v \le \lambda_v \le (1+\Delta')\cdot\hat{\lambda}_v$ holds with a probability of at least $1 - m\delta - me^{-\frac{(h+\eta)N}{8}}$.

For the heavy maximum matching phase and light maximum packing phase, since there is no requirement for the estimation, no additional analysis of the probability is needed.

Thus, by union bound over all four phases, we can summarize the probability mentioned in the statement. □

Appendix C: Missing proofs in Section 4

C.1. Proof of Proposition 1

PROPOSITION 1. *Denote* $N = \min_{v\in V} \lambda_v \cdot T$. *When* $h = 0$ *and* N *is large, for* $0 < \delta < 1$, *by setting* $\alpha = C_0 N^{-\frac{1}{3}}, \beta = 0.935\frac{2D}{2D+1}, \eta = \theta = \frac{2D}{2D+1}, \gamma = 0.084\frac{2D+1}{D^2}$, *where* C_0 *is a constant such that* α *satisfies* $3\sqrt{\frac{8ln\frac{1}{\delta}}{\alpha N}} = \alpha$, *with probability at least* $1 - m\delta - me^{\frac{-(h+\alpha)N}{8}}$, *the competitive ratio of Algorithm 1 is at least*

$$(1 - C_3 N^{-\frac{1}{3}}) \cdot \frac{e^{-0.225}}{4D+2},$$

where C_3 *is a specific constant defined in the proof.*

Proof We first assume $\eta = \theta$, i.e., no light LP phase. Then we choose $\alpha = C_0 N^{-\frac{1}{3}} \ll 1$ where C_0 is a constant that α satisfies $3\sqrt{\frac{8ln\frac{1}{\delta}}{\alpha N}} = \alpha$, i.e., $3\Delta = \alpha$. We rewrite the ratio $\min\{F^H, F^L\}$ in Theorem 1 where $F^H = \frac{1}{D}(1-3\Delta)(1-\alpha)(1-q_\alpha^\beta) + q_\alpha^\beta \frac{1}{D}\beta \ln\frac{\eta}{\beta}$, $F^L = \frac{q_\alpha^\beta \beta}{\eta}((1+2D)(1-\eta) + 2D\ln\eta)$ and $q_\alpha^\beta = e^{-D\gamma(1+\Delta)\frac{\beta-\alpha}{1-\alpha}}$. We set $\frac{e^{-\gamma\beta D}\beta}{\eta} = e^{-b}$, $\eta = \theta = \frac{2D}{2D+1}$ and $z = \gamma\beta D$ similar to Theorem 3 in Naori and Raz (2019), and we assume $\beta \gg \alpha$ (later we will easily verify this). To find a useful lower bound, we need to find out the order of infinitesimal terms (α), ignore these terms to tune the rest part, and finally add the infinitesimal terms back to get the lower bound.

First we find an approximation of q_α^β. $q_\alpha^\beta \approx e^{-z(1+\Delta)(1-\frac{\alpha}{\beta})(1+\alpha)} \approx e^{-z}(1 - z(\Delta + \alpha - \frac{1}{\beta}\alpha)) = e^{-z}(1 - C_1\alpha)$ where C_1 satisfies $C_1\alpha = z(\Delta + \alpha - \frac{1}{\beta}\alpha)$. These approximations are according to the infinitesimals Δ and α, and $e^{-x} \approx 1 - x$ when x is small.

Then $F^L \approx \frac{e^{-z}\beta}{\eta}(1 - C_1\alpha)((1+2D)(1-\eta) + 2D\ln\eta) = (1 - C_1\alpha) \cdot e^{-b}(1 - 2D\ln(1 + \frac{1}{2D}))$. This is because the choices of η and b. Then according to the fact that $\ln(1+x) \le x - \frac{1}{2}x^2 + \frac{1}{3}x^3$ when $x > 0$, we have $1 - 2D\ln(1 + \frac{1}{2D}) \ge \frac{1}{4D^2} - \frac{1}{12D^2}$. Further, it is easy to check that $\frac{1}{4D^2} - \frac{1}{12D^2} \ge \frac{1}{4D+2}$ when $D \ge 1$. Now we have

$$F^L \ge (1 - C_1\alpha) \cdot \frac{e^{-b}}{4D+2}.$$

Then we consider the approximation of $F^H = \frac{1}{D}(1-3\Delta)(1-\alpha)(1-q_\alpha^\beta) + q_\alpha^\beta \frac{1}{D}\beta\ln\frac{\eta}{\beta}$.

$$\begin{aligned} F^H &= \frac{1}{D}(1-3\Delta)(1-\alpha)(1-q_\alpha^\beta) + q_\alpha^\beta \frac{2}{2D+1} e^{-b} e^{z}(-z+b) \\ &\ge \frac{1}{D}(1-2\alpha)(1 - e^{-z}(1-C_1\alpha)) + (1-C_1\alpha)\frac{2}{2D+1} e^{-b}(-z+b). \\ &\ge (1 - C_2\alpha) \cdot \left[\frac{1}{D}(1-e^{-z}) + \frac{2}{2D+1} e^{-b}(-z+b)\right]. \end{aligned}$$

The first equation is because $\beta = \eta e^{-b} e^{z}$ and $\eta = \frac{2D}{2D+1}$, the first inequality is because $q_\alpha^\beta \approx e^{-z}(1 - C_1\alpha)$ and $(1-3\Delta)(1-\alpha) = (1-\alpha)^2 \ge 1 - 2\alpha$, and last inequality holds if we set $C_2 = 2 + C_1$.

Now considering F^L and F^H have the same order of infinitesimal term, i.e., $1 - C_1\alpha$ and $1 - C_2\alpha$, we ignore these infinitesimal terms and try to solve the following optimization problem:

$$\max_{z,b} \min\left\{\frac{e^{-b}}{4D+2}, \frac{1}{D}(1-e^{-z}) + \frac{2}{2D+1} e^{-b}(-z+b)\right\}.$$

Then we try to find z and the minimum b such that $\frac{1}{D}(1-e^{-z})+\frac{2}{2D+1}e^{-b}(-z+b)\geq\frac{e^{-b}}{4D+2}$ holds, which gives a lower bound of the optimization problem before. To get this, it suffices to show $e^{b}(1-e^{-z})+b-z\geq\frac{1}{4}$. We set $b=0.225$ and $z=0.158$, then $\beta=\frac{e^{-b}}{e^{-z}}\eta=0.935\eta=0.935\frac{2D}{2D+1}$ and $\gamma=\frac{z}{\beta D}=0.084\frac{2D+1}{D^2}$. We can summarize our final competitive ratio with infinitesimal term as

$$(1-C_3N^{-\frac{1}{3}})\cdot\frac{e^{-0.225}}{4D+2}$$

where $C_3=\max\{C_1,C_2\}\cdot C_0$. □

C.2. Proof of Proposition 2

PROPOSITION 2. *Denote $N=\min_{v\in V}\lambda_v\cdot T$. For $0<\delta<1$, by setting $\alpha=C_0N^{-\frac{1}{3}},\beta=\eta=\eta_1$, $\theta=1$, $\gamma=1$ and $\gamma'=\frac{1}{2D}$, where C_0 is a constant such that α satisfies $3\sqrt{\frac{8ln\frac{1}{\delta}}{\alpha N}}=\alpha$ and η_1 is the solution of $e^{D\eta}=\frac{5-\eta}{4}$, with a probability of at least $1-2m\delta-me^{\frac{-(h+\alpha)N}{8}}-me^{\frac{-(h+\eta)N}{8}}$, the competitive ratio of Algorithm 1 is at least*

$$\frac{1}{D}\cdot\frac{1-\eta_1}{5-\eta_1}\cdot(1-C_3N^{-\frac{1}{2}}(h+C_0N^{-\frac{1}{3}})^{-\frac{1}{2}}),$$

where C_3 is a specific constant defined in the proof.

Proof We consider the parameter settings that $\beta=\eta$ and $\theta=1$, i.e., there is no heavy maximum matching phase or light maximum packing phase. We choose $\alpha=C_0N^{-\frac{1}{3}}\ll 1$ where C_0 is a constant that α satisfies $3\sqrt{\frac{8ln\frac{1}{\delta}}{\alpha N}}=\alpha$. Now we rewrite the ratio $\min\{F^H,F^L\}$ in Theorem 1 where $F^H=\frac{1}{D}(1-3\Delta)(1-\alpha)(1-e^{-\gamma(1+\Delta)\frac{\eta-\alpha}{1-\alpha}D})$ and $F^L=e^{-\gamma(1+\Delta)\frac{\eta-\alpha}{1-\alpha}D}\cdot(1-2\Delta')\gamma'(1-\eta)(1-D\gamma'(1+\Delta'))$. Under the assumption that T is large, we ignore the high order of infinitesimal to make the expression clear.
For F^H, we first look at the term $(1-3\Delta)(1-\alpha)\approx 1-(3\Delta+\alpha)$. We can compare the order between the term Δ and α. If h is in an order higher than $N^{-\frac{1}{3}}$, we can approximate $3\Delta+\alpha$ by 3Δ, otherwise, we can find 3Δ and α is in the same order $N^{-\frac{1}{3}}$. Thus, to summarize these two cases, the term can be approximated by $1-C_1N^{-\frac{1}{2}}(h+C_0N^{-\frac{1}{3}})^{-\frac{1}{2}}$ where C_1 is a constant that satisfies $3\Delta+\alpha=C_1N^{-\frac{1}{2}}(h+C_0N^{-\frac{1}{3}})^{-\frac{1}{2}}$. For the term $e^{-\gamma(1+\Delta)\frac{\eta-\alpha}{1-\alpha}D}$, we can approximate it as $e^{-\gamma(1+\Delta)\frac{\eta-\alpha}{1-\alpha}D}\approx e^{-D\gamma(1+\Delta)(\eta-\alpha)(1+\alpha)}\approx e^{-D\gamma(\eta+\eta(\Delta+\alpha)-\alpha)}\approx e^{-\gamma\eta D}\cdot e^{-\gamma\eta D(\Delta+\alpha-\frac{\alpha}{\eta})}\approx e^{-\gamma\eta D}(1-\gamma\eta D(\Delta+\alpha-\frac{\alpha}{\eta}))$. These approximations are according to the infinitesimal Δ and α, and $e^{-x}\approx 1-x$ when x is small. Thus, $F^H\approx\frac{1}{D}(1-e^{-\gamma\eta D})(1-C_1N^{-\frac{1}{2}}(h+C_0N^{\frac{1}{3}})^{-\frac{1}{2}})$.
Similar for F^L, we can have the same order of the infinitesimal. Adopting $\Delta'\leq\Delta$, we can have a constant C_2 such that $F^L\geq e^{-\gamma\eta D}\cdot\gamma'(1-\eta)(1-D\gamma')(1-C_2N^{-\frac{1}{2}}(h+C_0N^{-\frac{1}{3}})^{-\frac{1}{2}})$. To find a near optimal choice of parameters, we ignore the infinitesimal term first. Then our problem is

$$\max_{\eta,\gamma,\gamma'}\min\left\{\frac{1}{D}(1-e^{-\gamma D\eta}),e^{-\gamma D\eta}\cdot\gamma'(1-\eta)(1-D\gamma')\right\}$$

Since γ' only influences F^L, we can choose $\gamma'=\frac{1}{2D}$ to maximize F^L. Then we can update the competitive ratio as $\max_{\eta,\gamma}\min\left\{\frac{1}{D}(1-e^{-\gamma D\eta}),\frac{1}{4D}e^{-\gamma D\eta}(1-\eta)\right\}$. Observing that $F^H=\frac{1}{D}(1-e^{-\gamma D\eta})$ and $F^L=\frac{1}{4D}e^{-\gamma D\eta}(1-\eta)$ are increasing with respect to γ, we choose $\gamma=1$. Then for the choice of η, since F^H increases with η while F^L decreases with η, we choose the η that satisfies $\frac{1}{D}(1-e^{-D\eta})=\frac{1}{4D}e^{-D\eta}(1-\eta)$. That is, $\eta=\eta_1$ satisfies $e^{D\eta_1}=\frac{5-\eta_1}{4}$. It is easy to check that there only exists one η_1 and this η_1 is feasible. Then the ratio is $\frac{1}{D}\cdot\frac{1-\eta_1}{5-\eta_1}$. Then add back the infinity small part and choose $C_3=\max\{C_1,C_2\}$, we have the wanted ratio. □

C.3. Proof of Proposition 3

To prove this proposition, we need the following two lemmas.

LEMMA 23. *Given $\eta \le 1-h$, we have competitive ratio of at least $\frac{1}{D}(h+\alpha)\ln\frac{h+\eta}{h+\alpha}$,when we set $\theta=\eta$, $\beta=\alpha$ and $\alpha=\max\left\{(h+\eta)e^{-\frac{Df_1}{h+\eta}}-h,(h+\eta)e^{-1}-h,0\right\}$. Here $f_1=(1-(h+\eta))(1+2D)+2D\ln(h+\eta)+h(1+2D\ln(h+\eta)-Dh)$.*

Proof We first assume that $F^H \ge F^L$, according to $\eta \le 1-h$, we have

$$F^H \ge F^L \iff \alpha \le \alpha_1 := (h+\eta)e^{-\frac{Df_1}{h+\eta}}-h.$$

When $\alpha_1 \ge 0$ and $\alpha \le \alpha_1$, the competitive ratio is $F^L=\frac{h+\alpha}{h+\eta}\cdot f_1$. This ratio is increasing with α then it is maximized when $\alpha=\alpha_1$. We know that when $\alpha=\alpha_1$, $F^L=F^H$, then the ratio can be written as $F^H=\frac{1}{D}(h+\alpha)\ln\frac{h+\eta}{h+\alpha}$. When $\alpha_1<0$, then we have $\alpha>\alpha_1$ which means $F^L\ge F^H$, then in this case we also have competitive ratio F^H. Thus, we only need to solve $\max_\alpha F^H$ when $\alpha \ge \max\{0,\alpha_1\}$. According to the derivative of F^H, we can easily see that when $\alpha \le \alpha_2=(h+\eta)e^{-1}-h$, the ratio is increasing while the ratio is decreasing when $\alpha\ge\alpha_2$. If $\alpha_2 \ge \max\{0,\alpha_1\}$, the optimal $\alpha=\alpha_2$. If $\alpha_2<\max\{0,\alpha_1\}$, then F^H is decreasing when $\alpha \ge \max\{0,\alpha_1\}$, so we choose optimal $\alpha=\max\{0,\alpha_1\}$.
Then to summarize, the optimal α is $\alpha=\max\left\{(h+\eta)e^{-\frac{Df_1}{h+\eta}}-h,(h+\eta)e^{-1}-h,0\right\}$ and the ratio is $\frac{1}{D}(h+\alpha)\ln\frac{h+\eta}{h+\alpha}$. □

LEMMA 24. *Given $\eta>1-h$, we have competitive ratio of at least*

$$\begin{cases}\frac{1}{D}(h+\alpha)(\ln\frac{1}{h+\alpha}+1-e^{1-h-\eta}), & \alpha_1\le 1-h\\ e^{1-h-\eta}\cdot f_2, & \alpha_1 \ge 1-h\end{cases}$$

when we set $\theta=\eta$, $\beta=\alpha$ and

$$\alpha=\begin{cases}\max\{\alpha_1,\alpha_2,0\}, & \alpha_1\le 1-h\\ 1-h, & \alpha_1\ge 1-h\end{cases}$$

Here $f_2=(1-\eta)(1-D(1-\eta))$, $\alpha_1=\exp\{1-(Df_2+1)e^{1-h-\eta}\}-h$ and $\alpha_2=\exp\{-e^{1-h-\eta}\}-h$.

Proof First recall the competitive ratio is $\min\{F^H,F^L\}$ where $F^H=\frac{1}{D}(h+\alpha)(\ln\frac{1}{h+\alpha}+1-e^{1-h-\eta})$, $F^L=(h+\alpha)e^{1-h-\eta}\cdot f_2$ and $f_2=(1-\eta)(1-D(1-\eta))$. We first assume that $F^H\ge F^L$, according to $\eta>1-h$ we have

$$F^H \ge F^L \iff \alpha\le\alpha_1:=\exp\{1-(Df_2+1)e^{1-h-\eta}\}-h.$$

When $\alpha_1\ge 0$ and $\alpha\le\alpha_1$, the competitive ratio is $F^L=(h+\alpha)e^{1-h-\eta}f_2$ which is increasing with α. If $\alpha_1 \ge 1-h\ge 0$, because we only consider $\alpha\le 1-h$, we always have $F^H\ge F^L$, then the ratio is F^L, and we choose $\alpha=1-h$. If $0\le\alpha_1\le 1-h$, we can choose $\alpha=\alpha_1$, and we have $F^L=F^H$. When $\alpha_1<0$, then we always have $\alpha>\alpha_1$ which means the competitive ratio can be written as F^H.
Now we only need to solve $\max_\alpha F^H$ when $\alpha\ge\max\{\alpha_1,0\}$ and $\max\{\alpha_1,0\}\le 1-h$. From the derivative of $F^H=\frac{1}{D}(h+\alpha)(\ln\frac{1}{h+\alpha}+1-e^{1-h-\eta})$, we can see that when $\alpha\le\alpha_2:=\exp\{-e^{1-h-\eta}\}-h$, F^H is increasing, and when $\alpha\ge\alpha_2$, F^H is decreasing. We can also have $\alpha_2\le 1-h$. Then, we have the following cases:

1. $\alpha_2\ge\max\{\alpha_1,0\}$: set $\alpha=\alpha_2=\max\{\alpha_1,\alpha_2,0\}$;

2. $\alpha_2 \le \max\{\alpha_1, 0\}$: set $\alpha = \max\{\alpha_1, 0\} = \max\{\alpha_1, \alpha_2, 0\}$.

By summarizing the cases above, we get our statement. □

We now come back to show the correctness of Proposition 3.

PROPOSITION 3. *Denote h_0 as $(2D^2+1)(\sqrt{1+\frac{1}{4D^2}}-1)+\frac{1}{2D}$. We can achieve the corresponding competitive ratios under different choices of h. Specifically,*

- *if $h \le \frac{1}{2D}$, by setting $\alpha = \beta = \frac{2D}{2D+1}(1+h)e^{-\frac{f_1(2D+1)}{2(1+h)}} - h$, $\eta = \theta = \frac{2D}{2D+1}(1+h) - h$, we achieve a competitive ratio of $f_1 e^{-\frac{f_1(2D+1)}{2(1+h)}}$. Here, $f_1 = 1 - 2Dh + 2D(1+h)\ln\left[\frac{2D(1+h)}{2D+1}\right] + h - Dh^2$;*
- *if $h \ge h_0$, by setting $\alpha = \beta = 1-h$, $\eta = \theta = 2 - \frac{1}{2D} - \sqrt{1+\frac{1}{4D^2}}$, we achieve a competitive ratio of $e^{1-h-\eta}(1-\eta)(1-D(1-\eta))$;*
- *if $\frac{1}{2D} < h < h_0$, by setting $\theta = \eta = 1 - \frac{1}{2D}$, $\beta = \alpha$ and*

$$\alpha = \begin{cases} \max\{\alpha_1, \alpha_2, 0\}, & \alpha_1 \le 1-h \\ 1-h, & \alpha_1 \ge 1-h, \end{cases}$$

where $\alpha_1 = \exp\{1 - \frac{5}{4}e^{\frac{1}{2D}-h}\} - h$ and $\alpha_2 = \exp\{-e^{\frac{1}{2D}-h}\} - h$, we achieve a competitive ratio of

$$\begin{cases} \frac{1}{D}(h+\alpha)(\ln\frac{1}{h+\alpha} + 1 - e^{\frac{1}{2D}-h}), & \alpha_1 \le 1-h \\ e^{\frac{1}{2D}-h}\frac{1}{4D}, & \alpha_1 \ge 1-h. \end{cases}$$

Proof We first consider the case when $h \le \frac{1}{2D}$. According to Lemma 23, we know how to find the optimal choice of α and the competitive ratio given $\eta \le 1-h$. Here we choose η that maximize $f_1 = (1-(h+\eta))(1+2D) + 2D\ln(h+\eta) + h(1+2D\ln(h+\eta) - Dh)$. By calculus, we can find out that $\eta = \frac{2D}{2D+1}(1+h) - h$. When $h \le \frac{1}{2D}$, we can verify that such η is no greater than $1-h$. Now we have $f_1 = 1 - 2Dh + 2D(1+h)\ln\left[\frac{2D(1+h)}{2D+1}\right] + h - Dh^2$. From Lemma 23, $\alpha = \max\left\{(h+\eta)e^{-\frac{Df_1}{h+\eta}} - h, (h+\eta)e^{-1} - h, 0\right\}$. Then we will show that $\alpha = (h+\eta)e^{-\frac{Df_1}{h+\eta}} - h$, i.e., $(h+\eta)e^{-\frac{Df_1}{h+\eta}} - h \ge (h+\eta)e^{-1} - h$ and $(h+\eta)e^{-\frac{Df_1}{h+\eta}} - h \ge 0$. To show $(h+\eta)e^{-\frac{Df_1}{h+\eta}} - h \ge 0$ it suffices to show that $(h+\eta)e^{-\frac{Df_1}{h+\eta}} \ge h$. Because $e^x > 1+x$, we only need to show that $h + \eta - Df_1 - h \ge 0$. Then given the value of η and f_1, we denote

$$g(h) = \eta - Df_1 = \frac{2D}{2D+1} - \frac{1}{2D+1}h + (2D^2 - D)h + D^2h^2 - D - 2D^2(1+h)\ln\left[\frac{2D(1+h)}{2D+1}\right].$$

Now we only need to show that $g(h) \ge 0$ for $0 \le h \le \frac{1}{2D}$. We check the first order derivative $g'(h)$ of $g(h)$: $g'(h) = -\frac{1}{2D+1} - D + 2D^2h - 2D^2\ln\left[\frac{2D(1+h)}{2D+1}\right]$, and the second order derivative $g''(h) = 2D^2(1 - \frac{1}{1+h}) \ge 0$. Then $g'(h) \le g'\left(\frac{1}{2D}\right) = -\frac{1}{2D+1} \le 0$ which means

$$\begin{aligned} g(h) \ge g\left(\frac{1}{2D}\right) &= \frac{2D}{2D+1} - \frac{1}{2D+1}\frac{1}{2D} + (2D^2 - D)\frac{1}{2D} + D^2\left(\frac{1}{2D}\right)^2 - D \\ &= \frac{1}{2D(2D+1)}\left[(2D)^2 - 1 + D(4D^2-1) + D^2\frac{2D+1}{2D} - D(2D+1)2D\right] \\ &= \frac{1}{2D(2D+1)}\left(3D^2 - \frac{1}{2}D - 1\right) \ge 0. \end{aligned}$$

Last inequality is because D is a positive integer. Now we have proved that $(h+\eta)e^{-\frac{Df_1}{h+\eta}} - h \ge 0$, then we try to prove that $(h+\eta)e^{-\frac{Df_1}{h+\eta}} - h \ge (h+\eta)e^{-1} - h$. It suffices to show that $-\frac{Df_1}{h+\eta} \ge -1$ which equals that $h + \eta - Df_1 \ge 0$. Because we have proved that $h + \eta - Df_1 - h \ge 0$ above, $h + \eta - Df_1 \ge 0$ already satisfies. Then the choice of α is $\alpha = (h+\eta)e^{-\frac{Df_1}{h+\eta}} - h = \frac{2D}{2D+1}(1+h)e^{-\frac{f_1(2D+1)}{2(1+h)}} - h$ and the competitive ratio is $\frac{1}{D}(h+\alpha)\ln\frac{h+\eta}{h+\alpha} = f_1 e^{-\frac{f_1(2D+1)}{2(1+h)}}$.

We then consider the second case where $h \geq h_0$. From Lemma 24, considering the case that $\alpha_1 \geq 1-h$, the optimal $\alpha = 1-h$, and competitive ratio is $e^{1-h-\eta}f_2$. This value is maximized when we choose $\eta = 2 - \frac{1}{2D} - \sqrt{1+\frac{1}{4D^2}}$. Then we have $f_2 = \sqrt{4D^2+1} - 2D$ and $\alpha_1 = \exp\{1-(Df_2+1)e^{1-h-\eta}\} - h$. It is easy to check that $\eta \geq 1-h$ because

$$\eta - (1-h) = h + 1 - \frac{1}{2D} - \sqrt{1+\frac{1}{4D^2}} \geq h_0 + 1 - \frac{1}{2D} - \sqrt{1+\frac{1}{4D^2}} = 2D^2\left(\sqrt{1+\frac{1}{4D^2}} - 1\right) \geq 0.$$

And $\eta \leq 1$ because $\eta - 1 = 1 - \frac{1}{2D} - \sqrt{1+\frac{1}{4D^2}} \leq -\frac{1}{2D} \leq 0$. The first inequality is because $\sqrt{1+\frac{1}{4D^2}} \geq 1$.
We have proved the feasibility of η, now we need to prove that $\alpha_1 \geq 1-h$. It suffices to show that $e^{h-(1-\eta)} \geq 1 + D(\sqrt{4D^2+1} - 2D)$. According to the fact $e^x > 1+x$, we only need to prove that $h+\eta \geq 1 + D(\sqrt{4D^2+1} - 2D)$. Then we need to prove that $h \geq 1 - \eta + D(\sqrt{4D^2+1} - 2D) = (2D^2+1)(\sqrt{1+\frac{1}{4D^2}} - 1) + \frac{1}{2D} = h_0$. The statement holds because of our assumption.
We can also see that $h_0 \leq 1$ because

$$h_0 = (2D^2+1)(\sqrt{1+\frac{1}{4D^2}} - 1) + \frac{1}{2D} \leq (2D^2+1)\frac{1}{8D^2} + \frac{1}{2D} = \frac{1}{4} + \frac{1}{2D} + \frac{1}{8D^2} \leq 1.$$

First inequality is because the fact $\sqrt{1+a} \leq 1 + \frac{1}{2}a$ for $a \geq 0$ and last inequality is obvious because D is a positive integer. According to Lemma 24, the competitive ratio is $e^{1-h-\eta}(1-\eta)(1-D(1-\eta))$.

For the last case where $\frac{1}{2D} < h < h_0$, observing the choices of η in the case when $h \geq h_0$, to ensure the optimal η which maximizes $e^{1-h-\eta}f_2$ satisfying the assumption $\eta > 1-h$, the condition that $h \geq h_0$ is necessary. Thus, for the case where $\frac{1}{2D} < h < h_0$, we choose the smallest η which keeps $\eta > 1-h$ below. We then adopt Lemma 24 and get our statement. □